Series 4 - Solutions

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4.13

Field 1: Since $u = (-V_0/\ell)x$ and $v = (V_0/\ell)y$,

it follows that streamlines are given by solution of the equation

$$\frac{dy}{dx} = \frac{v}{u} = \frac{(V_0/\ell)y}{-(V_0/\ell)x} = -\frac{y}{x}$$

in which variables can be separated and the equation integrated to give

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \Rightarrow \ln y = -\ln x + \text{constant}$$

Thus, along a streamline $xy = C$ where C is a constant

By using different values of the constant C, we can plot various lines in the x-y plane – the streamlines – as plotted in Fig. for $x \geq 0$.

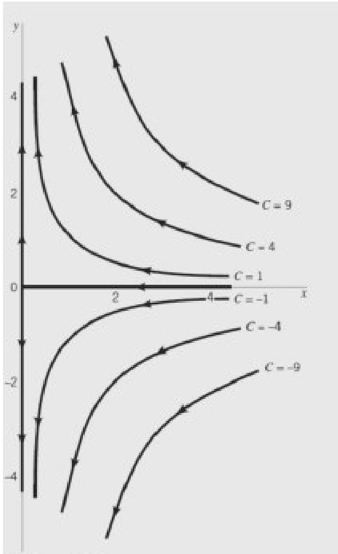

Figure E4.2

Field 2: Likewise, streamlines are given by

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{xy^2}{x^2y} = -\frac{y}{x} \quad \text{or} \quad \frac{dy}{y} = -\frac{dx}{x}$$ which can

be integrated as: $\int \frac{dy}{y} = -\int \frac{dx}{x}$

Thus, $\ln y = -\ln x + \tilde{c}$ where $\tilde{c}$ is a constant

Thus, $xy = c'$

Comment: These streamlines are the same shape (same "flow pattern") as in the Field 1 – but the velocity fields are different!

The flow is not completely specified by the shape of the streamlines alone

4.31 As a valve is opened, water flows through the diffuser shown in Fig. P4.31 at an increasing flowrate so that the velocity along the centerline is given by $\mathbf{V} = u\hat{i} = V_0(1-e^{-ct})(1-x/\ell)\hat{i}$, where u_0, c, and ℓ are constants. Determine the acceleration as a function of x and t. If $V_0 = 3.0$ m/s and $\ell = 1.5$ m, what value of c (other than $c=0$) is needed to make the acceleration zero for any x at $t = 1$ s? Explain how the acceleration can be zero if the flowrate is increasing with time.

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V}$$

with $u = u(x,t)$, $v = 0$, and $w = 0$, this becomes

$$\vec{a} = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\hat{i} = a_x\hat{i}, \quad \text{where } u = V_0(1-e^{-ct})(1-\frac{x}{\ell})$$

Thus,

$$a_x = V_0\left(1-\frac{x}{\ell}\right)c\,e^{-ct} + V_0^2(1-e^{-ct})^2\left(1-\frac{x}{\ell}\right)\left(-\frac{1}{\ell}\right)$$

or

$$\underline{\underline{a_x = V_0\left(1-\frac{x}{\ell}\right)\left[c\,e^{-ct} - \frac{V_0}{\ell}(1-e^{-ct})^2\right]}}$$

If $a_x = 0$ for any x at $t = 1$ s, we must have $\left[c\,e^{-ct} - \frac{V_0}{\ell}(1-e^{-ct})^2\right] = 0$ with $V_0 = 3\,\frac{m}{s}$ and $\ell = 1.5$ m

$$c\,e^{-c} - \frac{3}{1.5}(1-e^{-c})^2 = 0.$$

The solution (root) of this equation is $\underline{c = 0.5\ \frac{1}{s}}$

For the above conditions the local acceleration ($\frac{\partial u}{\partial t} > 0$) is precisely balanced by the convective deceleration ($u\frac{\partial u}{\partial x} < 0$). The flowrate increases with time, but the fluid flows to an area of lower velocity.

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Notes on how to solve

$$f(c) = ce^{-c} - 2(1-e^{-c})^2 = 0 \quad (1)$$

This quadratic polynomial contains a non-linear term (ce^{-c}) which means that it can not be solved analytically.

One obvious root is, $c=0$.
In order to find the second root, we can employ a method of numerical analysis, e.g. the iteration method.

We can write (1) as: $ce^{-c} - 2(1-2e^{-c}+e^{-2c}) = 0$

$$\Rightarrow \quad -2e^{-2c} + (4+c)e^{-c} - 2 = 0$$

which gives $e^{-c} = \dfrac{-(4+c) \pm \sqrt{(4+c)^2-16}}{-4}$ let's consider only the (+) root

hence $c = -\ln\left[\dfrac{-(4+c)+\sqrt{(4+c)^2-16}}{-4}\right]$

Let $g(c) = -\ln\left[\dfrac{-(4+c)+\sqrt{(4+c)^2-16}}{-4}\right] \quad (2)$

thus, $c = g(c) \quad (3)$

Let $c = c_0 = 1$ be an initial approximation of the required root.
Then from (2), we calculate $g(c=1) = 0.69$. But since $c = g(c)$ from (3) our initial approximation was wrong, with an error $= c - c_0 = 0.31$.
We repeat this process for the renewed value, $c = c_1$ and so on, until the error becomes small. After 12 iteration steps, the method converges to $c \cong 0.5$.

step	c	g(c)	error
1	1.0000	0.6931	3.07E-01
2	0.6931	0.5805	1.13E-01
3	0.5805	0.5324	4.81E-02
4	0.5324	0.5104	2.20E-02
5	0.5104	0.5000	1.05E-02
6	0.5000	0.4949	5.04E-03
7	0.4949	0.4925	2.45E-03
8	0.4925	0.4913	1.20E-03
9	0.4913	0.4907	5.86E-04
10	0.4907	0.4904	2.87E-04
11	0.4904	0.4902	1.41E-04
12	0.4902	0.4902	6.90E-05

4.36

4.36 A nozzle is designed to accelerate the fluid from V_1 to V_2 in a linear fashion. That is, $V = ax + b$, where a and b are constants. If the flow is constant with $V_1 = 10$ m/s at $x_1 = 0$ and $V_2 = 25$ m/s at $x_2 = 1$ m, determine the local acceleration, the convective acceleration, and the acceleration of the fluid at points (1) and (2).

With $u = ax + b$, $v = 0$, and $w = 0$ the acceleration $\vec{a} = \frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V}$ can be written as

$\vec{a} = a_x \hat{i}$ where $a_x = u\frac{\partial u}{\partial x}$. (1)

Since $u = V_1 = 10\frac{m}{s}$ at $x = 0$ and $u = V_2 = 25\frac{m}{s}$ at $x = 1$ we obtain

$10 = 0 + b$

$25 = a + b$ so that $a = 15$ and $b = 10$

That is, $u = (15x + 10)\frac{m}{s}$, where $x \sim m$, so that from Eq.(1)

$a_x = (15x+10)\frac{m}{s}\left(15\frac{1}{s}\right) = \underline{\underline{(225x+150)\frac{m}{s^2}}}$

Note: The local acceleration is zero, $\underline{\underline{\frac{\partial \vec{V}}{\partial t} = 0}}$, and the convective acceleration is $u\frac{\partial u}{\partial x}\hat{i} = \underline{\underline{(225x+150)\hat{i}\,\frac{m}{s^2}}}$

At $x = 0$, $\vec{a} = \underline{\underline{150\,\hat{i}\,\frac{m}{s^2}}}$; at $x = 1\,m$, $\vec{a} = \underline{\underline{375\,\hat{i}\,\frac{m}{s^2}}}$

4.52

4.52 Water flows steadily through the funnel shown in Fig. P4.52. Throughout most of the funnel the flow is approximately radial (along rays from O) with a velocity of $V = c/r^2$, where r is the radial coordinate and c is a constant. If the velocity is 0.4 m/s when $r = 0.1$ m, determine the acceleration at points A and B.

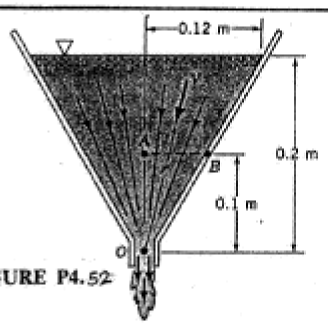


FIGURE P4.52

$\vec{a} = a_n \hat{n} + a_s \hat{s}$, where $a_n = \frac{V^2}{\mathcal{R}} = 0$ since $\mathcal{R} = \infty$ (i.e., the streamlines are straight)

Also, $a_s = V\frac{\partial V}{\partial s} = V\frac{\partial V}{\partial r}$, where $V = \frac{c}{r^2}$

Since $V = 0.4\,\frac{m}{s}$ when $r = 0.1m$ it follows that

$c = Vr^2 = (0.4\,\frac{m}{s})(0.1\,m)^2 = 4\times10^{-3}\,\frac{m^3}{s}$, or $V = \frac{4\times10^{-3}}{r^2}\,\frac{m}{s}$, where $r \sim m$

Thus,

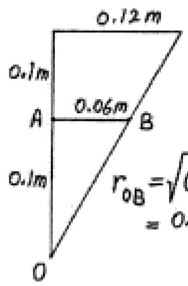
$$a_s = -\left(\frac{c}{r^2}\right)\left(-\frac{2c}{r^3}\right) = \frac{2c^2}{r^5}$$

At point A:

$$a_s = \frac{2(4\times10^{-3}\,\frac{m^3}{s})^2}{(0.1m)^5} = \underline{\underline{3.20\,\frac{m}{s^2}}}$$

At point B:

$$a_s = \frac{2(4\times10^{-3}\,\frac{m^3}{s})^2}{(0.1167m)^5} = \underline{\underline{1.48\,\frac{m}{s^2}}}$$



$r_{OB} = \sqrt{(0.1)^2 + (0.06)^2} = 0.1167m$

4.54 Air flows from a pipe into the region between two parallel circular disks as shown in Fig. P4.54. The fluid velocity in the gap between the disks is closely approximated by $V = V_0 R/r$, where R is the radius of the disk, r is the radial coordinate, and V_0 is the fluid velocity at the edge of the disk. Determine the acceleration for $r = 0.3$, 0.6 or 0.9 if $V_0 = 1.5$ m/s and $R = 0.9$ m.

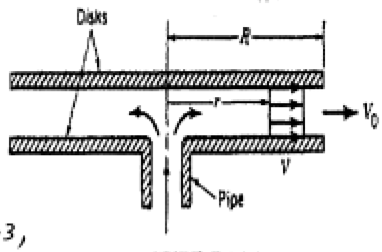


FIGURE P4.54

$\vec{a} = a_n \hat{n} + a_s \hat{s}$, where $a_n = \dfrac{V^2}{R} = 0$ since $R = \infty$

(i.e., the streamlines are straight)

Also, $a_s = V \dfrac{\partial V}{\partial s} = V \dfrac{\partial V}{\partial r}$, where $V = \dfrac{V_0 R}{r}$

Since $V_0 = 1.5 \frac{m}{s}$ and $R = 0.9$ m, $V = \dfrac{1.35}{r} \dfrac{m}{s}$, where $r \sim m$

Thus,

$$a_s = \left(\frac{V_0 R}{r}\right)\left(-\frac{V_0 R}{r^2}\right)$$

$$= -\frac{V_0^2 R^2}{r^3}$$

$$= -\frac{\left(1.5 \frac{m}{s}\right)^2 (0.9\,m)^2}{r^3\ m^3} = \frac{-1.82}{r^3} \frac{m}{s^2}, \quad r \sim m$$

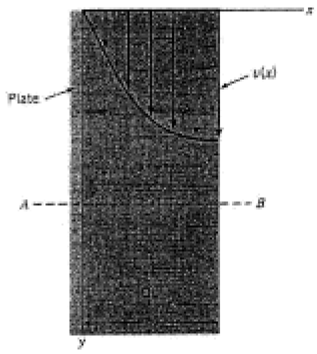
At $r = 0.3$ m, $a_s = -67.4 \frac{m}{s^2}$

At $r = 0.6$ m, $a_s = -8.42 \frac{m}{s^2}$

At $r = 0.9$ m, $a_s = -2.5 \frac{m}{s^2}$

4.68

4.68 A layer of oil flows down a vertical plate as shown in Fig. P4.68 with a velocity of $\mathbf{V} = (V_0/h^2)(2hx - x^2)\hat{\mathbf{j}}$ where V_0 and h are constants. (a) Show that the fluid sticks to the plate and that the shear stress at the edge of the layer ($x = h$) is zero. (b) Determine the flowrate across surface AB. Assume the width of the plate is b. (Note: The velocity profile for laminar flow in a pipe has a similar shape. See Video V6.13)



■ FIGURE P4.68

a) $v = \frac{V_0}{h^2}(2hx - x^2)$

Thus,

$$v\Big|_{x=0} = \frac{V_0}{h^2}(0-0) = 0 \quad \text{and}$$

$$\tau\Big|_{x=h} = \mu \frac{dv}{dx}\Big|_{x=h} = \mu \frac{V_0}{h^2}\left[2h - 2x\right|_{x=h} = 0$$

Hence, the fluid sticks to the plate and there is no shear stress at the free surface.

b) $$Q_{AB} = \int v\, dA = \int_{x=0}^{x=h} v\, b\, dx = \int_0^h \frac{V_0}{h^2}(2hx - x^2)\, b\, dx$$

or

$$Q_{AB} = \frac{V_0 b}{h^2}\left[hx^2 - \frac{1}{3}x^3\right|_0^h = \underline{\underline{\frac{2}{3}V_0 h b}}$$

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