Fisher Information Matrix

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Fisher Information Matrix

The Fisher Information Matrix (FIM) quantifies the amount of information a sample provides about parameters in a model.

- For a parameter vector $\boldsymbol{\theta}$, the Fisher Information Matrix $\mathrm{I}(\boldsymbol{\theta})$ is defined as:

$$\mathcal{I}(\boldsymbol{\theta}) = -\mathbb{E}\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top}\right]$$

  where $\ell(\boldsymbol{\theta})$ is the log-likelihood.
- Intuitively, $\mathcal{I}(\boldsymbol{\theta})$ measures how sensitive the likelihood is to changes in $\boldsymbol{\theta}$.

Calculation of the Fisher Information in Regression

In linear regression, the Fisher Information Matrix for parameter vector $\boldsymbol{\beta}$ is calculated based on the model:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \epsilon, \ \epsilon \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$$

For this model,

- The log-likelihood is:

$$\ell(\boldsymbol{\beta}) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mathbf{x}_i^\top\boldsymbol{\beta})^2$$

- Taking second derivatives with respect to $\boldsymbol{\beta}$ gives:

$$\mathcal{I}(\boldsymbol{\beta}) = \frac{1}{\sigma^2}\mathbf{X}^\top\mathbf{X}$$

Interpreting the Fisher Information Matrix

The Fisher Information Matrix informs model evaluation and parameter reliability:

- **Variance of Estimates**: A larger Fisher Information indicates more precise estimates.
- **Parameter Sensitivity**: High sensitivity in $\ell(\beta)$ to changes in β implies higher information content.
- **Goodness of Fit**: Information metrics help assess model fit, as more information implies a better fit to observed data.