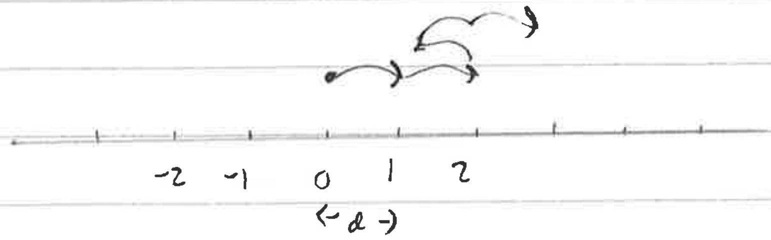
1D Discrete Random Walk

10/3/25



Let a walker start at $x = 0$, and take a sequence of steps of length d – moving right with probability p, and left with $1-p$.

What is its mean position after N steps?

$$X_N = N_R \cdot (+d) + N_L (-d)$$

N_R = # steps right
N_L = # steps left

$$= N_R \cdot d + (N - N_R) \cdot -d$$

$$= (2N_R - N)\, d$$

What is the probability of this position?

$$P(N_R, N_L; N) = \frac{N!}{N_R!\, N_L!}\, p^{N_R} (1-p)^{N_L}$$

Let $n = N_R$ for convenience, then:

$$\langle X_N \rangle = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!}\, p^n (1-p)^{N-n} \, (2n - N)\, d$$

$$\Rightarrow \langle X_N \rangle = d(1-p)^N \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} \underbrace{\left(\frac{p}{1-p}\right)^n}_{x^n} (2n-N)$$

Let $x = \frac{p}{1-p}$

$$\Rightarrow \langle X_N \rangle = d(1-p)^N \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} x^n (2n-N)$$

$$= d(1-p)^N \left[2 \underbrace{\sum n x^n \frac{N!}{n!(N-n)!}}_{x \frac{d}{dx} S_N(x)} - N \underbrace{\sum x^n \frac{N!}{n!(N-n)!}}_{S_N(x)} \right]$$

Recall: $S_N(x) = (1+x)^N = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} x^n$

and consider: $x \frac{d}{dx} S_N(x) = x \cdot N(1+x)^{N-1} = x \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} n x^{n-1}$

$$\Rightarrow x N (1+x)^{N-1} = \sum \frac{N!}{n!(N-n)!} n x^n$$

$$\Rightarrow \langle X_N \rangle = d(1-p)^N \left[2 x N (1+x)^{N-1} - N (1+x)^N \right]$$

$$= N d (1-p)^N (1+x)^{N-1} \left[2x - (1+x) \right]$$

$$= N d (1-p)^N (1+x)^{N-1} (x-1)$$

$$\Rightarrow \langle X_N \rangle = N \cdot d \, (1-p)^N \cdot \left(1 + \frac{p}{1-p}\right)^{N-1} \left(\frac{p}{1-p} - 1\right)$$

$$= N \cdot d \cdot \cancel{(1-p)^N} \cdot \left(\frac{1}{\cancel{1-p}}\right)^{N-1} \cdot \left(\frac{p-(1-p)}{\cancel{1-p}}\right)$$

$$\underline{\underline{\langle X_N \rangle = N d \, (2p-1)}}$$

check,

1) $p=1$, i.e. all steps to right $\Rightarrow \langle X_N \rangle = N d$

2) $p=0$, i.e. all steps to left $\Rightarrow \langle X_N \rangle = -N d$

3) $p = \frac{1}{2}$, i.e. symmetric $\Rightarrow \langle X_N \rangle = 0$

<u>What is the variance?</u> $\mathrm{Var}(X_N) = \langle X_N^2 \rangle - \langle X_N \rangle^2$

$$\langle X_N^2 \rangle = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} \; d^2 (2n-N)^2$$

$$= d^2 (1-p)^N \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} x^n \left(4n^2 - 4nN + N^2\right)$$

and now we have three related summations;

Given $S_N(x) = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} x^n = \underline{(1+x)^N}$

and $x\frac{dS_N}{dx} = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} n x^n = \underline{Nx(1+x)^{N-1}}$

then $\left(x\frac{d}{dx}\right)^2 S_N = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} n^2 x^n = x\left[N(1+x)^{N-1} + N(N-1)x.(1+x)^{N-2}\right]$

$= N.x.(1+x)^{N-2}\left[1 + \cancel{x} + (N-\cancel{1})x\right]$

$= \underline{N(Nx+1)x(1+x)^{N-2}}$

$\therefore \langle x_N^2 \rangle = d^2(1-p)^N\left[4.N(Nx+1)x(1+x)^{N-2} - 4N.Nx(1+x)^{N-1} + N^2.(1+x)^N\right]$

$= d^2(1-p)^N(1+x)^{N-2}\left[4xN(Nx+1) - 4xN^2(1+x) + N^2(1+x)^2\right]$

$=$ " $\left[\cancel{4N^2x^2} + 4Nx - 4N^2x - \cancel{4N^2x^2} + N^2(1+x)^2\right]$

$=$ " $\left[4Nx - 4N^2x + N^2 + 2N^2x + N^2x^2\right]$

$=$ " $\left[4Nx - 2N^2x + N^2 + N^2x^2\right]$

$=$ " $\left[4Nx + N^2(1-x)^2\right]$

$$\Rightarrow \langle x_N^2 \rangle = d^2 (1-p)^N \left(1 + \frac{p}{1-p}\right)^{N-2} \left[4N \frac{p}{1-p} + N^2 \left(1 - \frac{p}{1-p}\right)^2 \right]$$

$$= d^2 (1-p)^2 \left[\frac{4Np}{1-p} + N^2 \left(\frac{1-2p}{1-p} \right)^2 \right]$$

$$\langle x_N^2 \rangle = d^2 \left[4Np(1-p) + N^2 (1-2p)^2 \right]$$

$$\therefore \mathrm{Var}(x_N) = \langle x_N^2 \rangle - \langle x_N \rangle^2$$

$$= d^2 \left[4Np(1-p) + N^2 (1-2p)^2 \right] - N^2 d^2 (2p-1)^2$$

$$= 4Nd^2 p(1-p)$$

check;

1) $p=1$ or $p=0$ $\mathrm{Var} = 0$ as expected.

2) $p = 1/2$, $\mathrm{Var}(x_N) = Nd^2$, which is the largest variance.