# Einstein's Derivation of Brownian Motion

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- each particle executes motion independently of all others; and its movements in different time intervals are independent processes provided the intervals are not too small.

- consider a time interval $\tau$, which is small compared to observation time intervals, but still large enough that the particle's motion in two successive time intervals $\tau$ are independent of each other

Consider $n$ particles suspended in a liquid.
In an interval $\tau$, the $x$ coordinate of a particle will change by an amount $\Delta$, where $\Delta$ is different for each particle and equally likely to be + or −.

Let the number of particles whose $x$ coordinate change by $\Delta$ in $\tau$ be

$$dn = n.\phi(\Delta).d\Delta \qquad \text{with} \quad \phi(-\Delta) = \phi(\Delta)$$

$$\int_{-\infty}^{+\infty} \phi(\Delta)\,d\Delta = 1$$

$\phi(\Delta)$ different from zero only if $\Delta \sim 0$

Let $f(x,t)$ be the number of particles per unit volume at $(x,t)$.
The distribution at $t+\tau$ is

$$f(x,t+\tau)\,dx = dx\int_{-\infty}^{+\infty} f(x-\Delta,t)\,\phi(\Delta)\,d\Delta$$

Integral Eqn for Distribution

But $\tau$ is small so: $f(x,t+\tau) \sim f(x,t) + \tau\dfrac{\partial f}{\partial t} + \cdots$

and because $\phi(\Delta)$ is zero except for small $\Delta$:

$$f(x-\Delta,t) \sim f(x,t) - \Delta\frac{\partial f}{\partial x} + \frac{\Delta^2}{2!}\frac{\partial^2 f}{\partial x^2} + \cdots$$

Substituting this into the integral equation:

$$\left[f(x,t) + \tau\frac{\partial f}{\partial t} + \cdots\right]dx = dx\left[\int \left[f - \Delta\cdot\frac{\partial f}{\partial x} + \frac{\Delta^2}{2!}\cdot\frac{\partial^2 f}{\partial x^2} + \cdots\right]\phi(\Delta)\,d\Delta\right]$$

$$= dx\left[\int f(x,t)\,\phi(\Delta)\,d\Delta - \int \Delta\frac{\partial f}{\partial x}\phi(\Delta)\,d\Delta + \int\frac{\Delta^2}{2!}\frac{\partial^2 f}{\partial x^2}\phi(\Delta)\,d\Delta\right]$$

$\phi$ symmetric

$$\left[\cancel{f} + \tau\frac{\partial f}{\partial t}\right]\cancel{dx} = \cancel{dx}\left[\cancel{f}\underbrace{\int_{-\infty}^{+\infty}\phi(\Delta)\,d\Delta}_{1} - \frac{\partial f}{\partial x}\underbrace{\cancel{\int_{-\infty}^{+\infty}\Delta\,\phi(\Delta)\,d\Delta}}_{0} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\int_{-\infty}^{+\infty}\Delta^2\phi(\Delta)\,d\Delta\right]$$

$$\therefore \frac{\partial f}{\partial t} = \underbrace{\frac{1}{2\tau}\int_{-\infty}^{+\infty}\Delta^2\phi(\Delta)\,d\Delta}_{D}\cdot\frac{\partial^2 f}{\partial x^2} = D\frac{\partial^2 f}{\partial x^2}$$

DIFFUSION EQUATION

## Asymmetric Case

Einstein's derivation assumes $\phi(\Delta)$ is symmetric.
What happens if we release this assumption?

Let $\phi(\Delta) = \phi_S(\Delta) + \phi_{AS}(\Delta)$ where $\phi_S(-\Delta) = \phi_S(\Delta)$

$\phi_{AS}(-\Delta) = -\phi_{AS}(\Delta)$

$$f(x, t+\tau)\,dx = dx \int_{-\infty}^{+\infty} f(x-\Delta, t)\left(\phi_S + \phi_{AS}\right) d\Delta$$

$$= dx\left[\int_{-\infty}^{+\infty} f(x-\Delta, t)\,\phi_S\, d\Delta + \int_{-\infty}^{+\infty} f(x-\Delta, t)\,\phi_{AS}\, d\Delta\right]$$

Again, expanding on $\tau$ and $\Delta$:

$$\left[\cancel{f} + \tau\frac{\partial f}{\partial t} + \cdots\right]\cancel{dx} = \cancel{dx}\left[\int\left[f(x,t) - \Delta\frac{\partial f}{\partial x} + \frac{\Delta^2}{2!}\frac{\partial^2 f}{\partial x^2} + \cdots\right]\phi_S\, d\Delta\right.$$

$$\left. + \int\left[f(x,t) - \Delta\frac{\partial f}{\partial x} + \frac{\Delta^2}{2!}\frac{\partial^2 f}{\partial x^2}\right]\phi_{AS}\, d\Delta\right]$$

$$= \int_{-\infty}^{+\infty} \cancel{f}\,\underbrace{\phi_S\, d\Delta}_{=1} - \frac{\partial f}{\partial x}\underbrace{\int_{-\infty}^{+\infty}\Delta\,\phi_S\, d\Delta}_{=0} + \frac{1}{2\tau}\frac{\partial^2 f}{\partial x^2}\underbrace{\int_{-\infty}^{+\infty}\Delta^2\phi_S\, d\Delta}_{D}$$

$$+ \underbrace{\int_{-\infty}^{+\infty} f\,\phi_{AS}\, d\Delta}_{=0} - \frac{\partial f}{\tau\,\partial x}\underbrace{\int_{-\infty}^{+\infty}\Delta\,\phi_{AS}\, d\Delta}_{A} + \frac{1}{2\tau}\frac{\partial^2 f}{\partial x^2}\underbrace{\int_{-\infty}^{+\infty}\Delta^2\phi_{AS}\, d\Delta}_{=0}$$

$$\therefore \frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2} - A \frac{\partial f}{\partial x}$$

FOKKER - PLANCK EQN WITH CONSTANT DRIFT & DIFFUSION.