Bioimage Informatics

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Course

Introduction to Fourier Analysis

Continuous Fourier Transform

1D

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)\, e^{-j\,\omega\, x}\, \mathrm{d}x$$

$$\hat{f}(\omega) = \mathscr{F}\{(x)\}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega)\, e^{j\,\omega\, x}\, \mathrm{d}\omega$$

$$f(x) = \mathscr{F}^{-1}\{\hat{f}(w)\}$$

Reminder

- Unit imaginary number

  $j = \sqrt{-1} = e^{j\pi/2}$
- ω is in radian
- Euler formula

  $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

nD

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{x})\, e^{-j\,\langle \boldsymbol{\omega} \boldsymbol{x}\rangle}\, \mathrm{d}x_1 \ldots \mathrm{d}x_d$$

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\omega})\, e^{j\,\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle}\, \mathrm{d}\omega_1 \ldots \mathrm{d}\omega_d$$

Vectorial notation

$$\boldsymbol{x} = (x_1, \ldots\, x_d) \in \mathbb{R}^d$$

$$\boldsymbol{\omega} = (\omega_1, \ldots\, \omega_d) \in \mathbb{R}^d$$

$$\langle \boldsymbol{\omega}, \boldsymbol{x} \rangle = \omega_1 x_1 + \ldots + \omega_d x_d \in \mathbb{R}$$

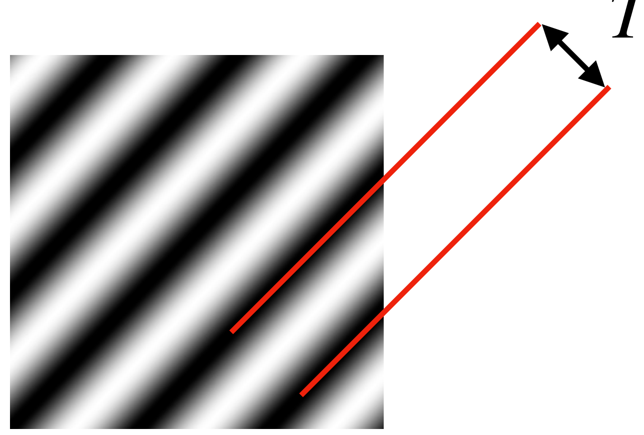
2D

$$\hat{f}(\omega_x, \omega_y) = \int f(x, y)\, e^{-j(\omega_x x + \omega_y y)}\, \mathrm{d}x\mathrm{d}y$$

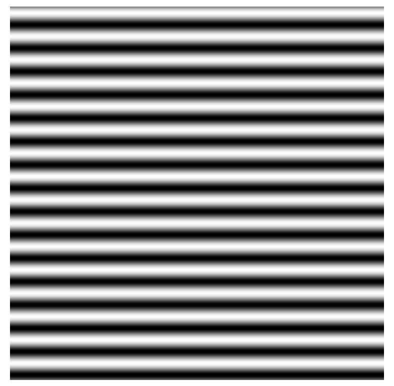
$$f(x, y) = \frac{1}{(2\pi)^2} \int \hat{f}(\omega_x, \omega_y)\, e^{j\,(\omega_x x + \omega_y y)}\, \mathrm{d}\omega_x \mathrm{d}\omega_y$$

Plane waves

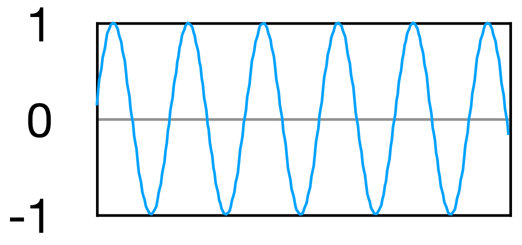
$$T = \frac{2\pi}{\sqrt{\omega_x^2 + \omega_y^2}}$$



Varying phase



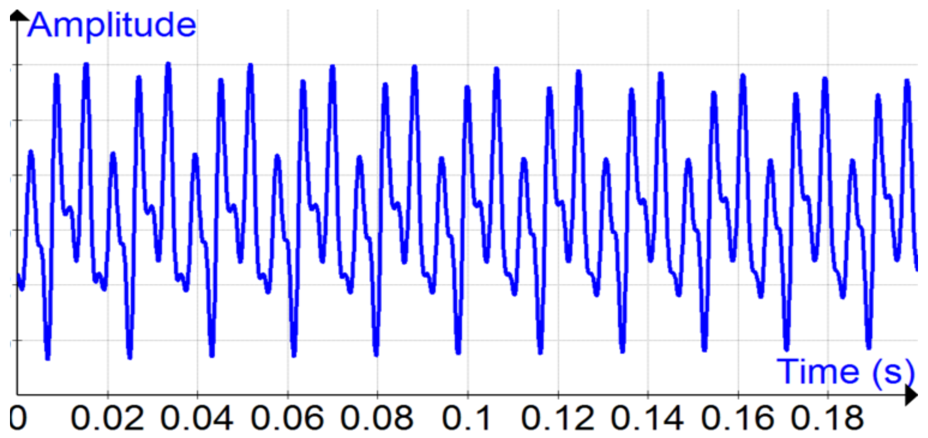
Varying direction

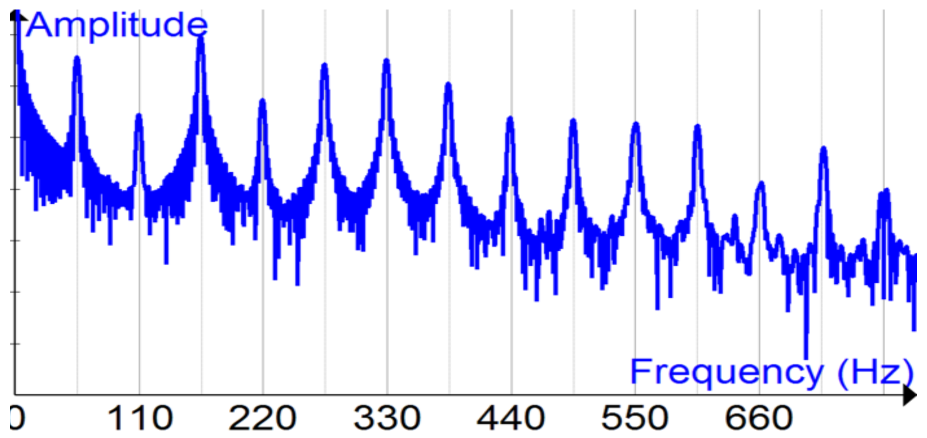


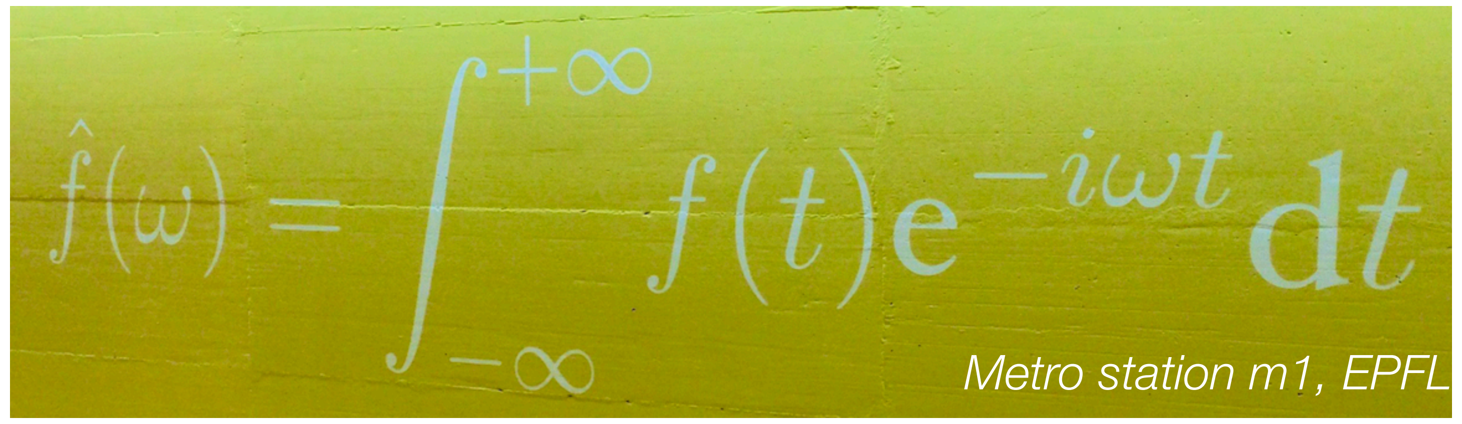
Fourier Analysis

The Fourier Transform can represent a signal by an infinite sum of sine and cosine waves.

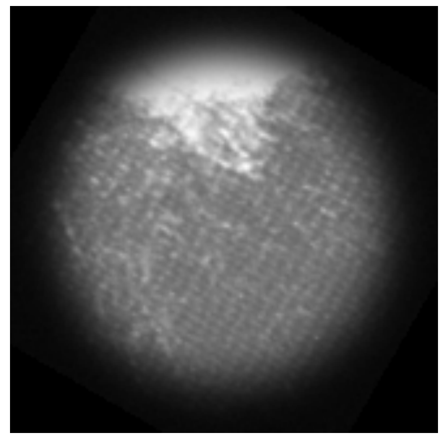
Harmonic Analysis

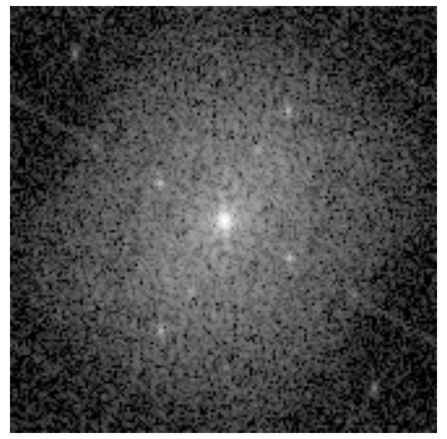




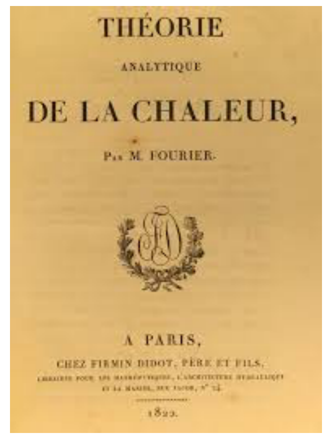


Metro station m1, EPFL





Frequency in 2D?



High impact in science and tech.

- Math. tool for the harmonic analysis
- Telecommunication
- Signal and image processing
- Imaging: interferometry, MRI, SIM

FFT Fast Fourier Transform

- Computation of the DFT O(N log(N))
- Cooley–Tukey algo, faster for radix-2
- Separable, GPU implementation
- FFTW: Fastest FFT in the West

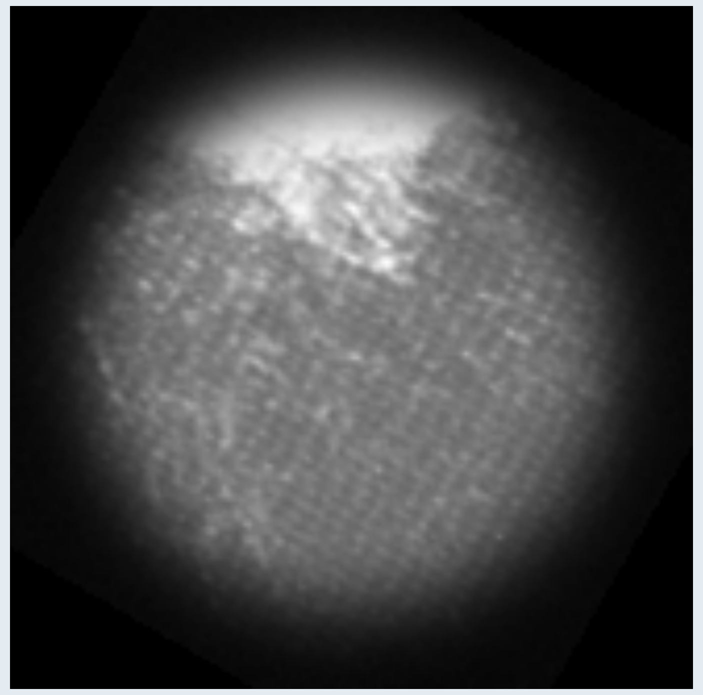
Historical notes

1822 Jean Baptiste Joseph Fourier (FR)
1910 Michel Plancherel (CH)
1965 James Cooley, John Tukey (USA)

Discrete Fourier Transform

Direct transform $\hat{f}[u,v] = \sum_{k} \sum_{l} f[k,l]\, e^{-j\frac{2\pi}{N}(uk+vl)}$

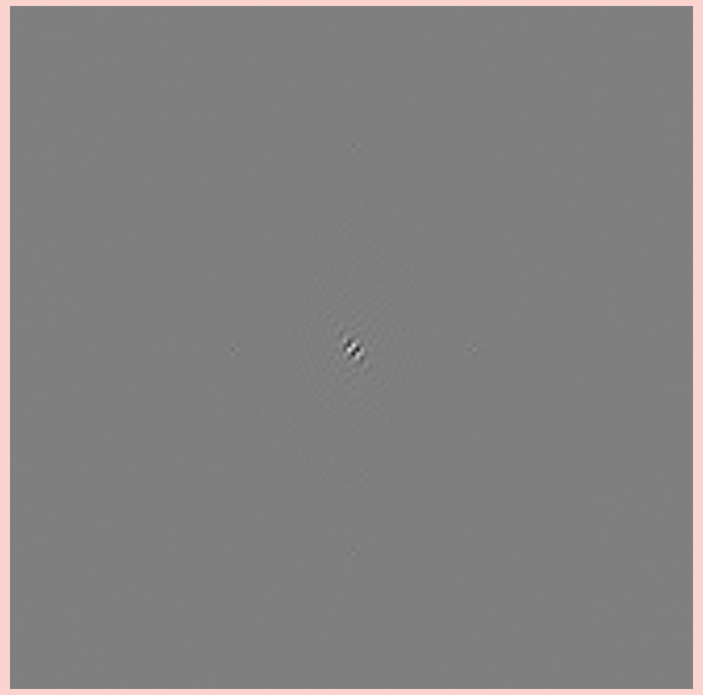
Space domain
real/imaginary



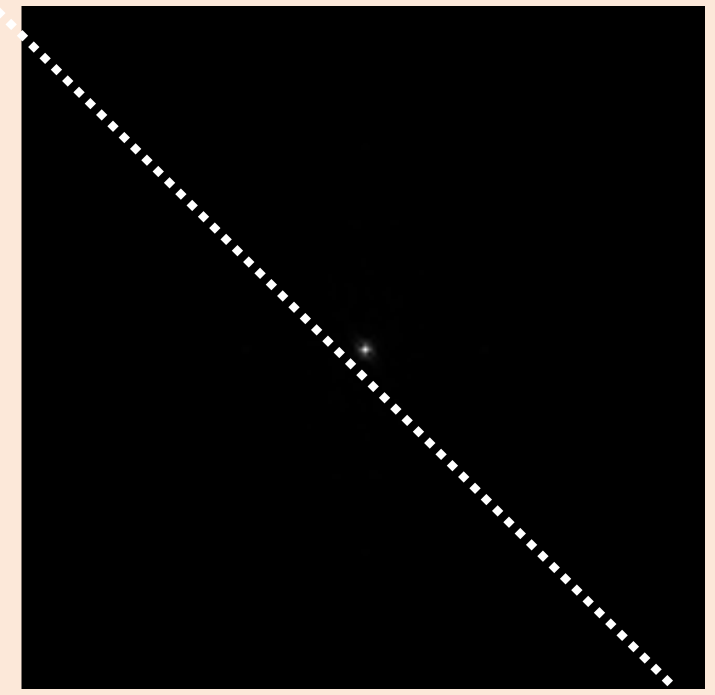
$Im\{f[k,l]\} = 0$

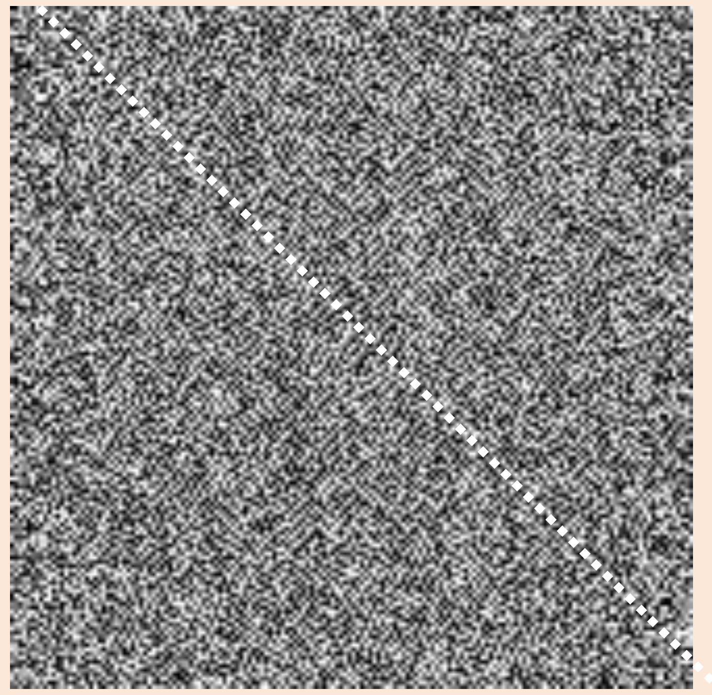
direct transform
analysis

Fourier domain
real/imaginary



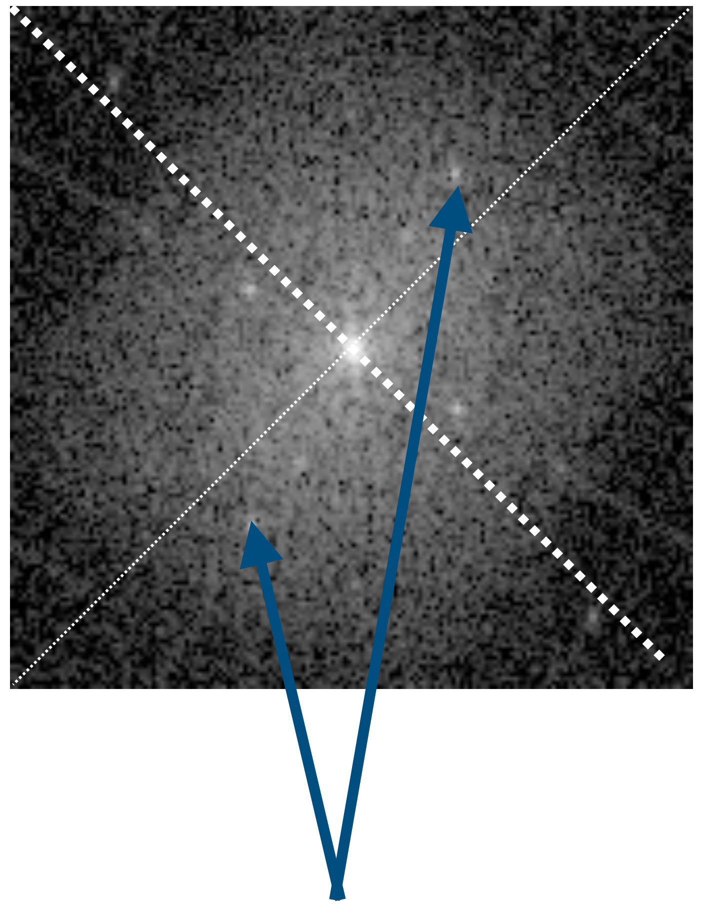
Fourier domain
modulus/phase





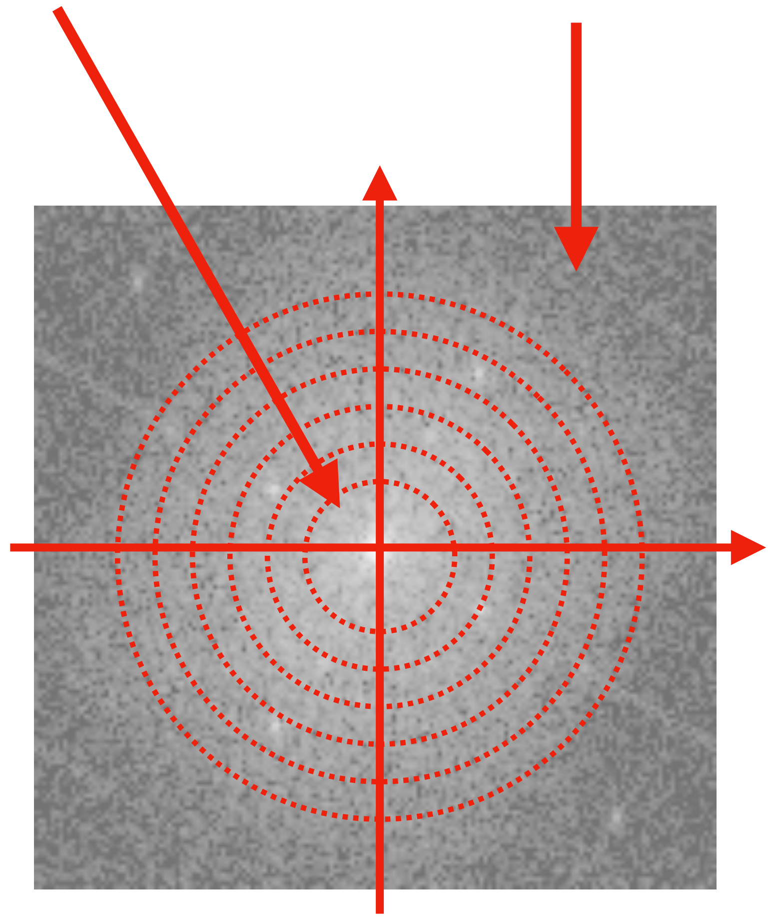
Phase [-π, π]

log



low frequencies

high frequencies



Hermitian symmetry

$f[k,l]$ **real** $\Longleftrightarrow$

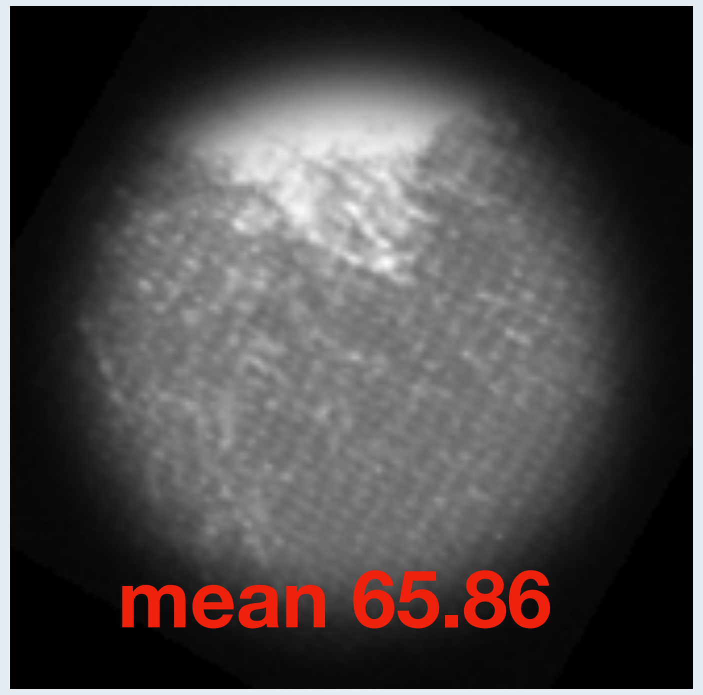
$\hat{f}[u,v] = \hat{f}^*[-u,-v]$

Discrete Fourier Transform

Direct transform $\hat{f}[u,v] = \sum_k \sum_l f[k,l]\, e^{-j\frac{2\pi}{N}(uk+vl)}$

Inverse transform $f[k,l] = \frac{1}{N^2} \sum_u \sum_v \hat{f}[u,v]\, e^{j\frac{2\pi}{N}(uk+vl)}$

Space domain
real/imaginary

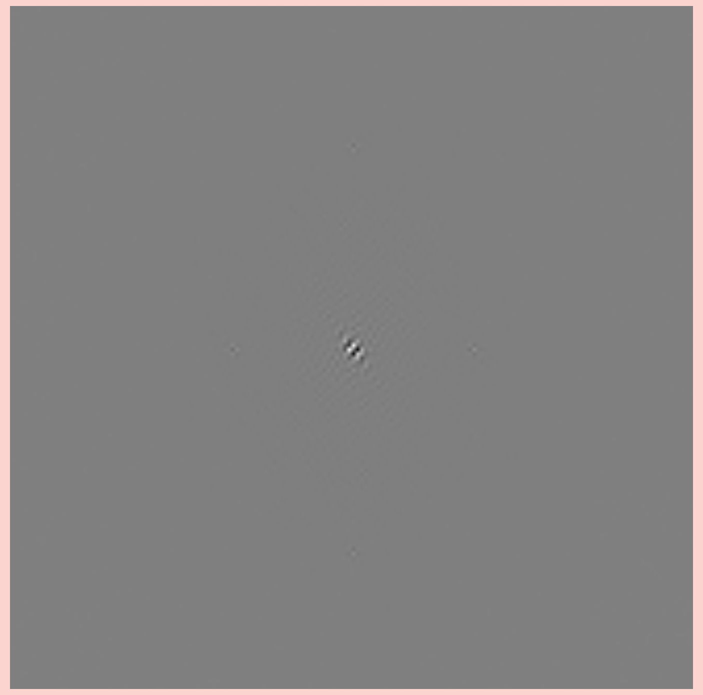


$Im\{f[k,l]\} = 0$

mean 0

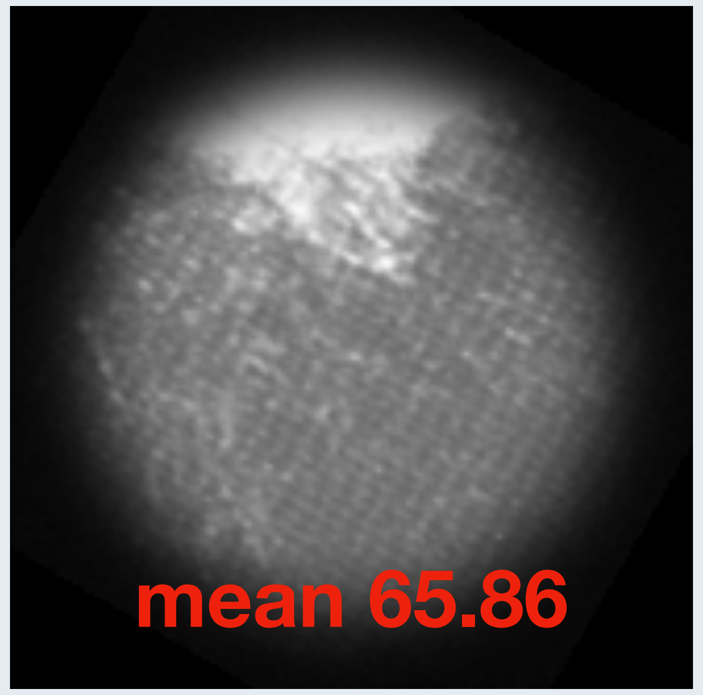
direct transform
analysis

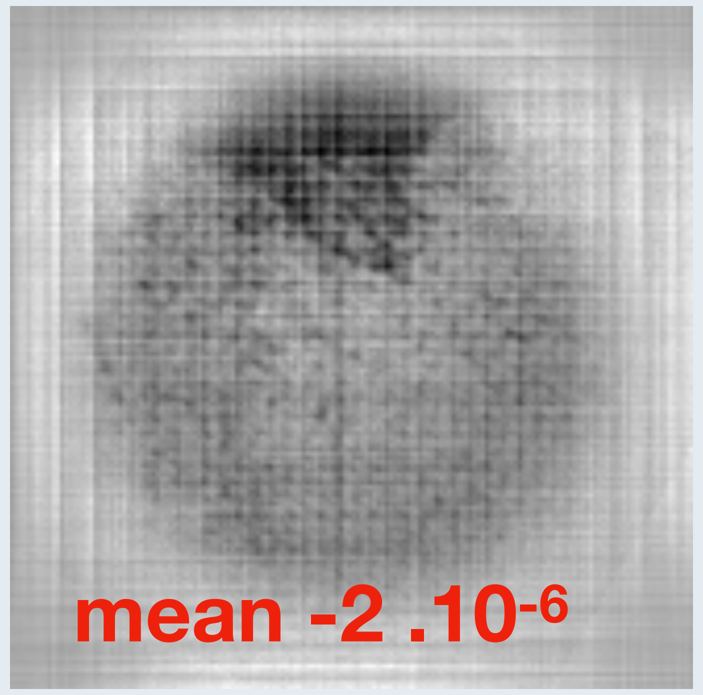
Fourier domain
real/imaginary



inverse transform
synthesis

Space domain
real/imaginary



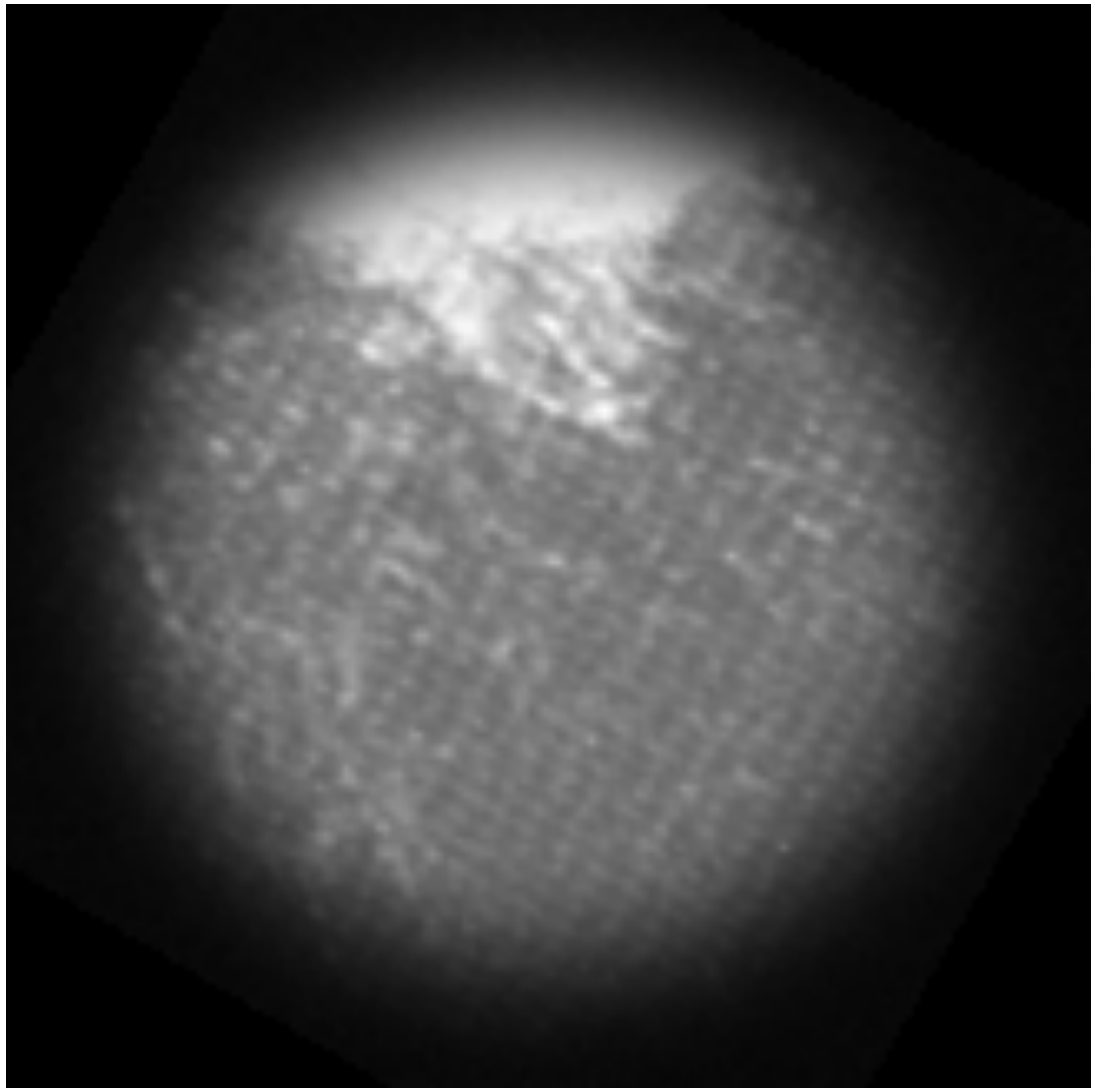


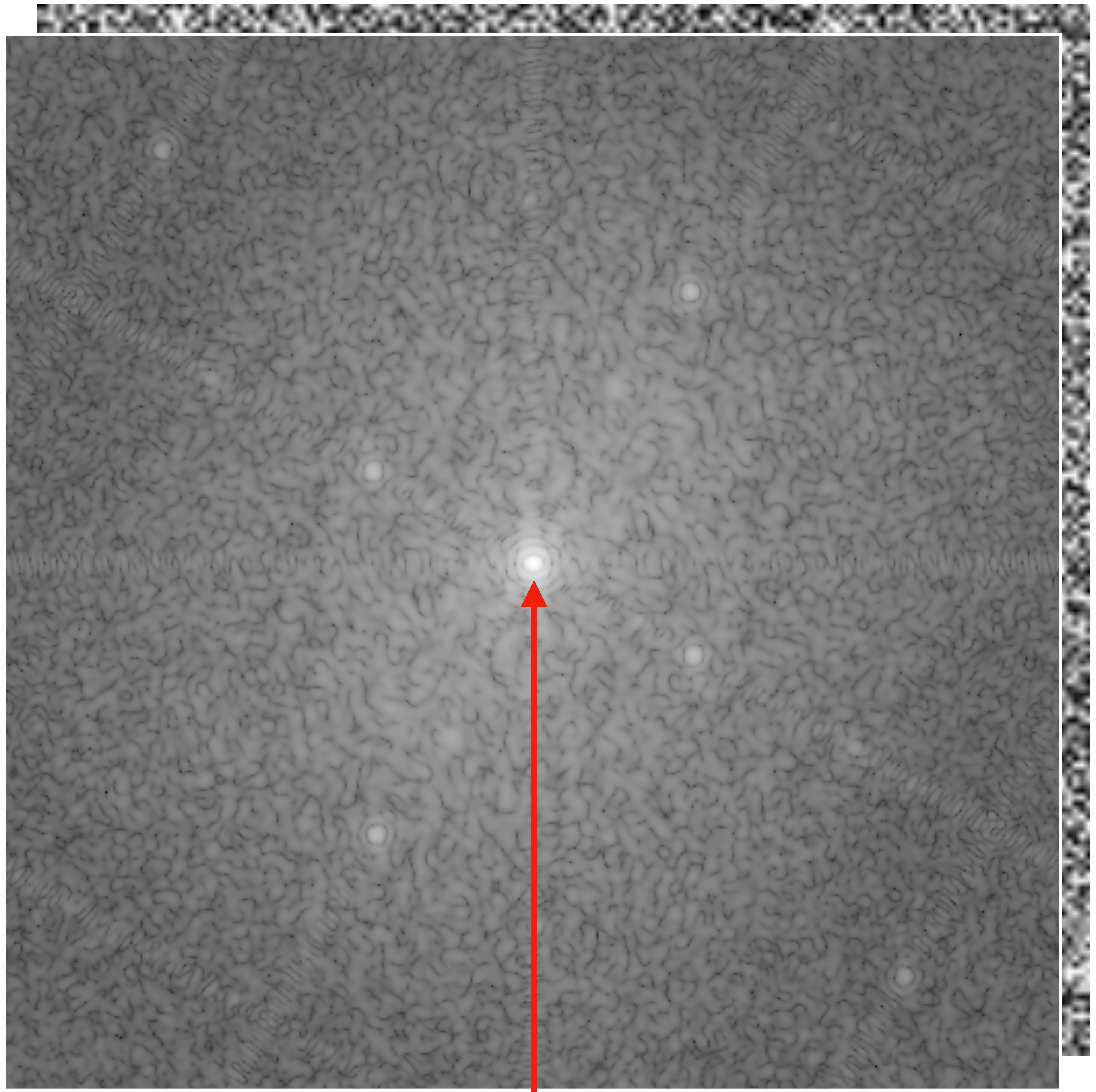
Perfect Reconstruction

- Up to the numerical errors
- Energy conservation
- Parseval identity

$$\int |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \propto \int |\hat{f}(\boldsymbol{\omega})|^2 \mathrm{d}\boldsymbol{\omega}$$

Interpretation of Frequencies



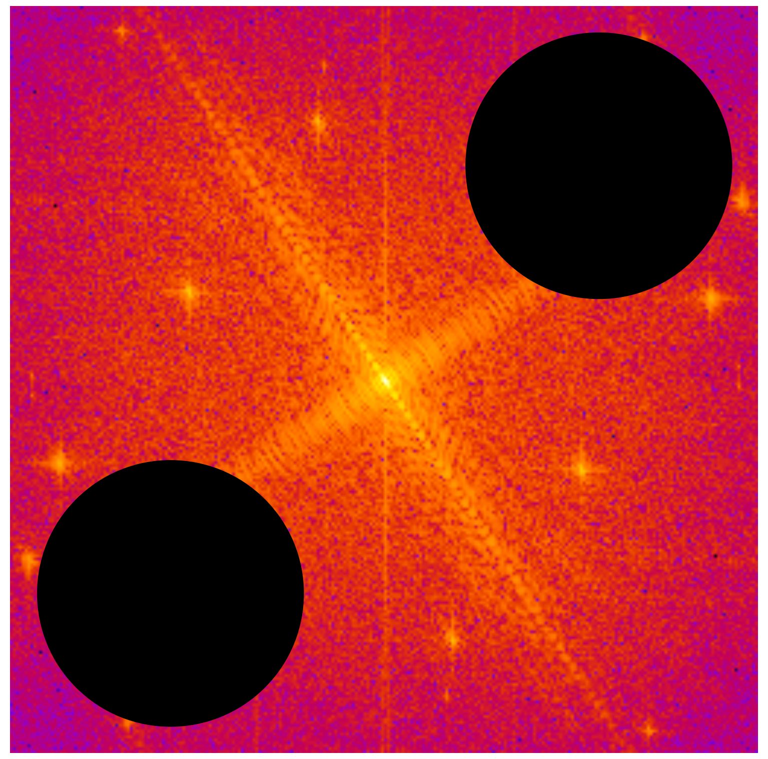


DC coef. $\hat{f}[0,0] = \sum \sum f[k,l]$

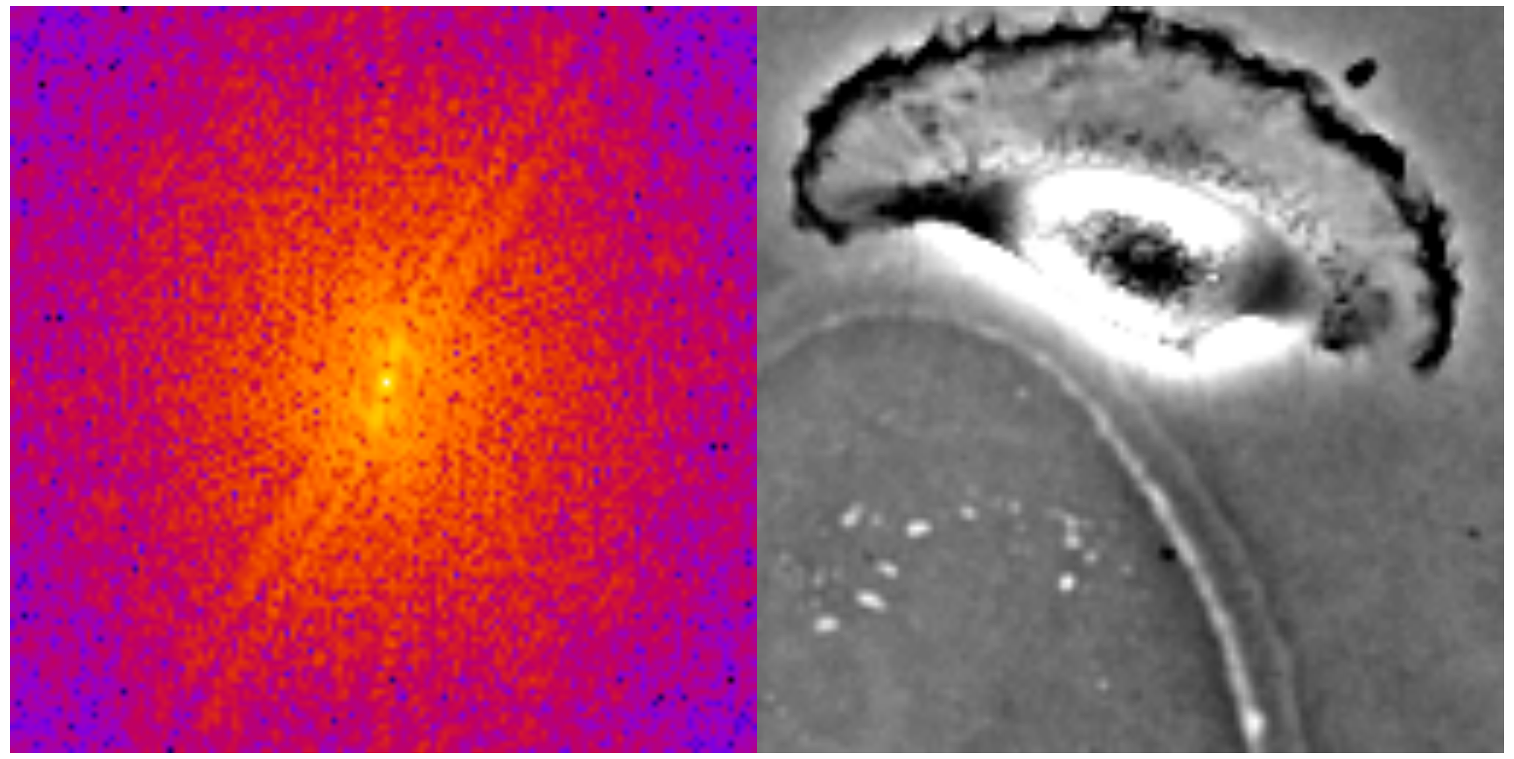
Questions

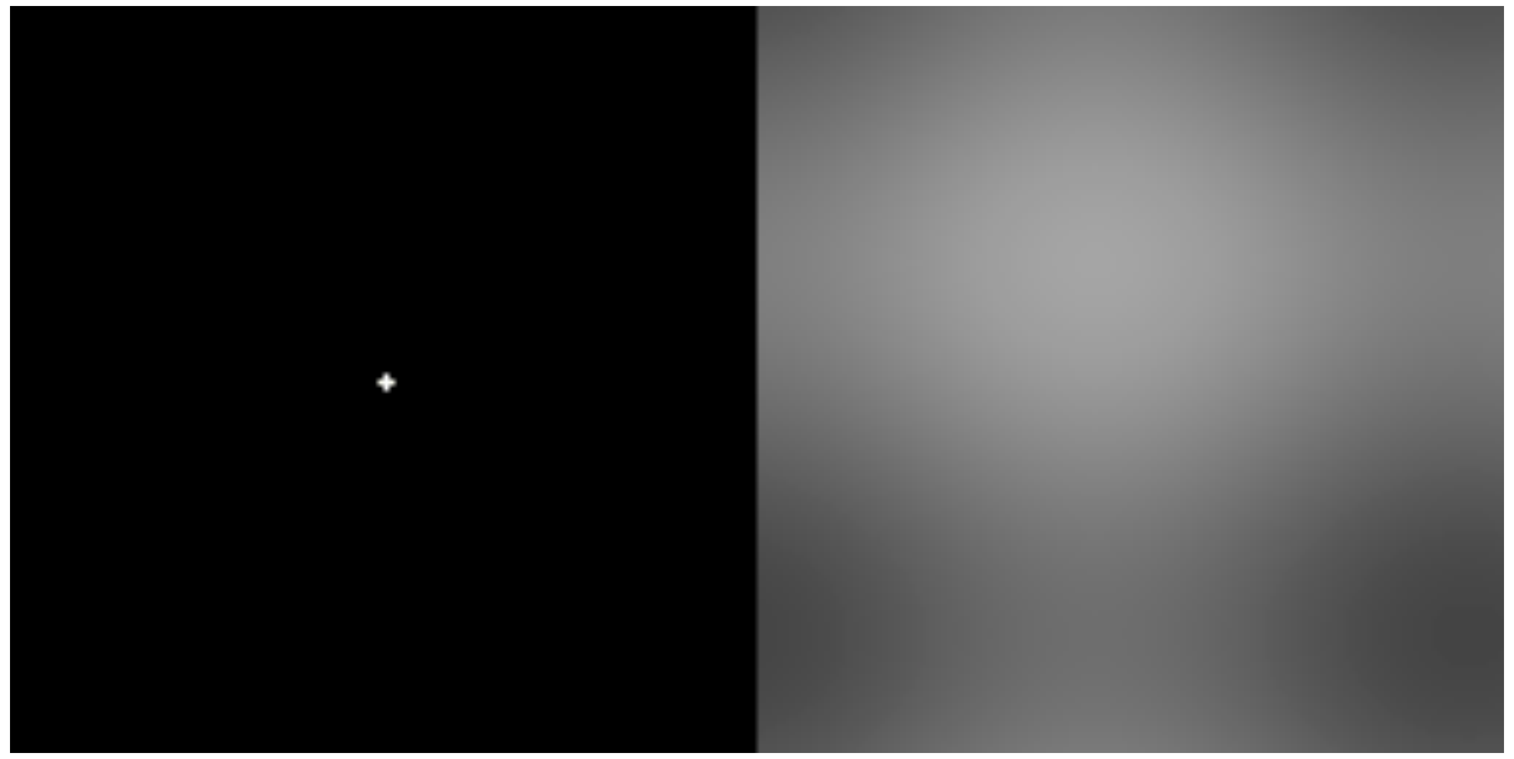
- How to qualify the quality of the frequency content?
- How to observe the Hermitian symmetry?
- What are the 4 bright spots
- What is the central coefficient
- How to compute the step size of the grid?
- How to remove the grid pattern from the original image?

Partial Reconstruction

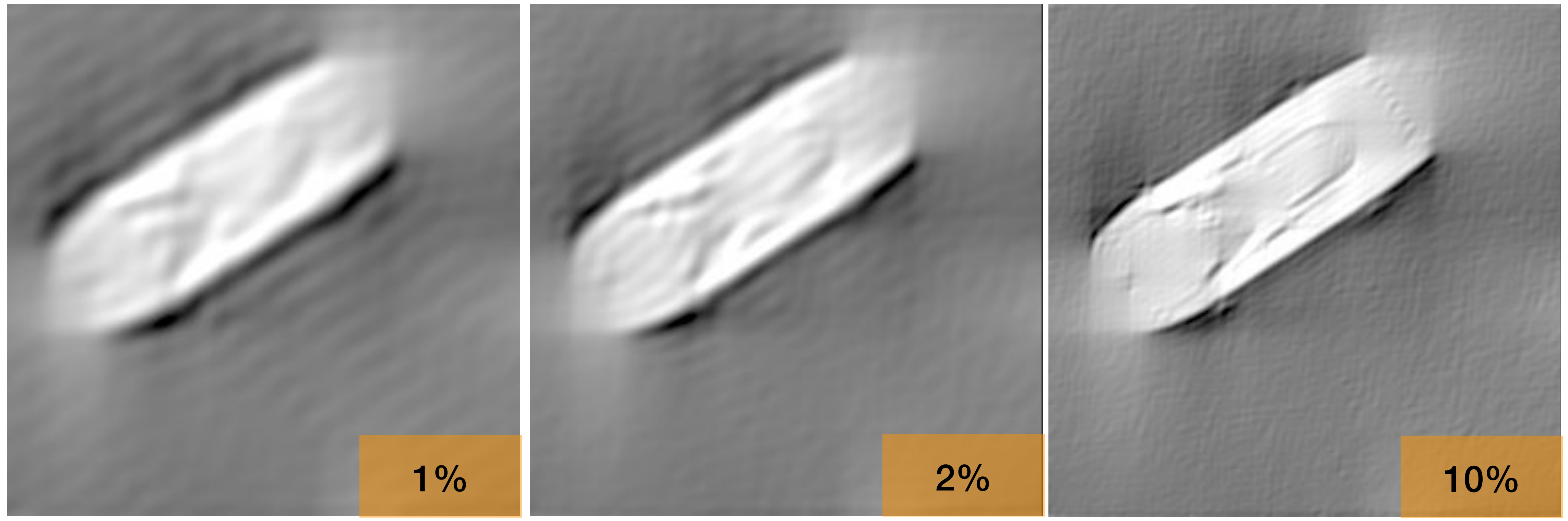


Partial reconstruction
- Set some coef. to 0
- Keep intact the NZ

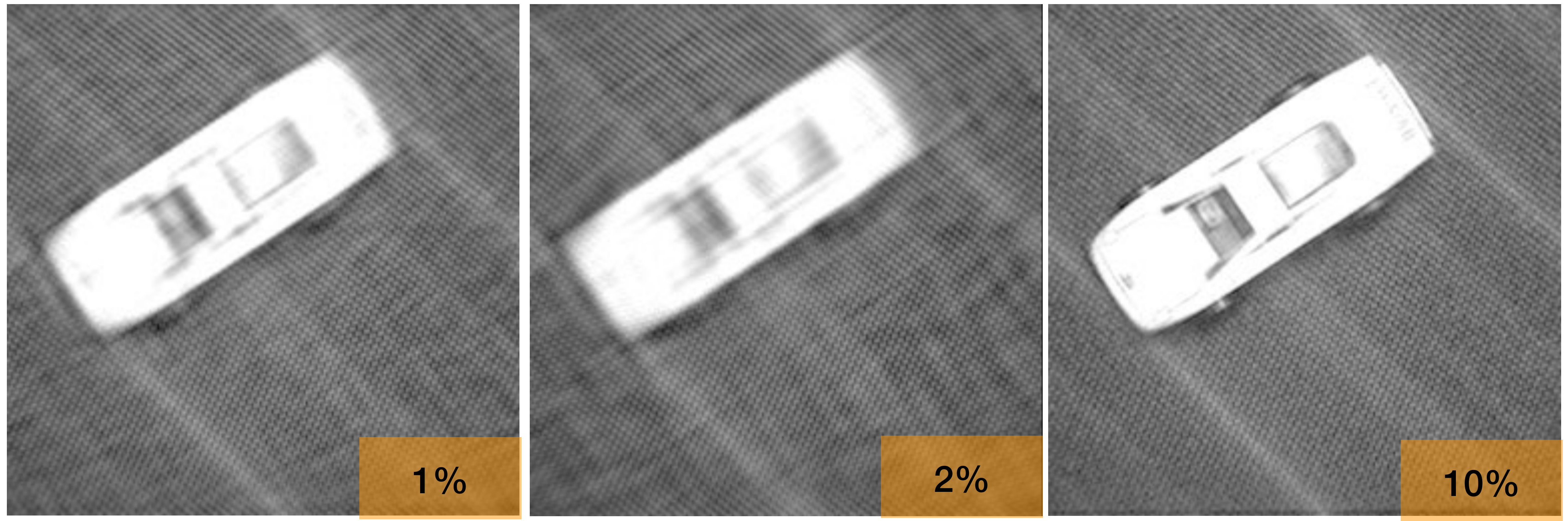


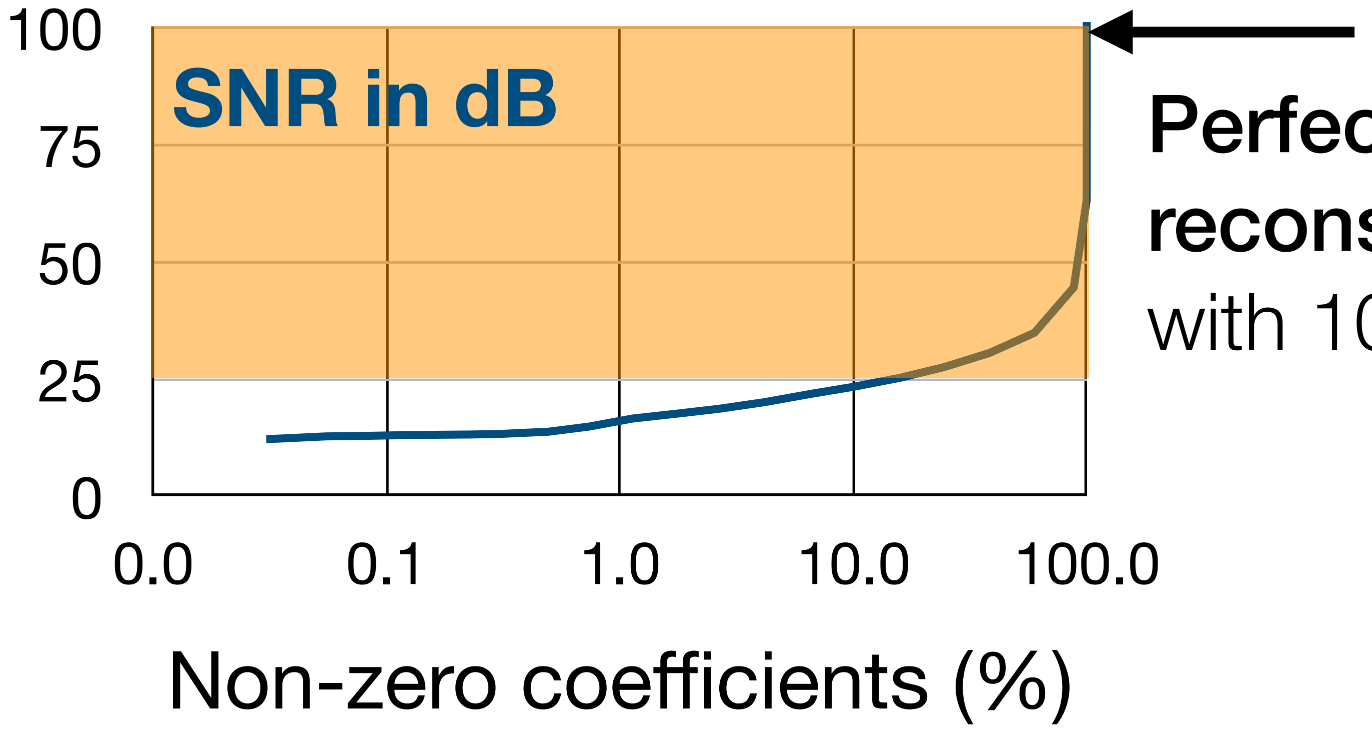


Keep only lowest frequencies



Keep only largest coefficients



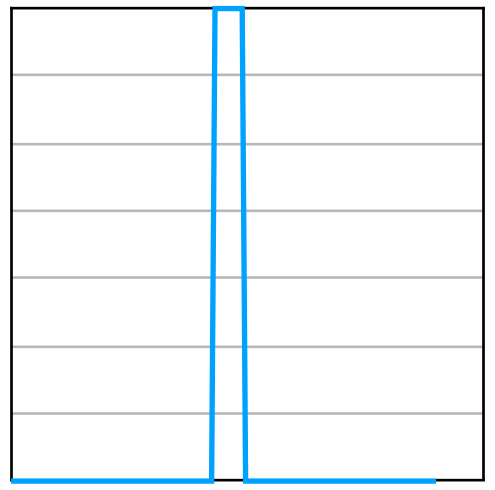


Perfect reconstruction with 100% coef.

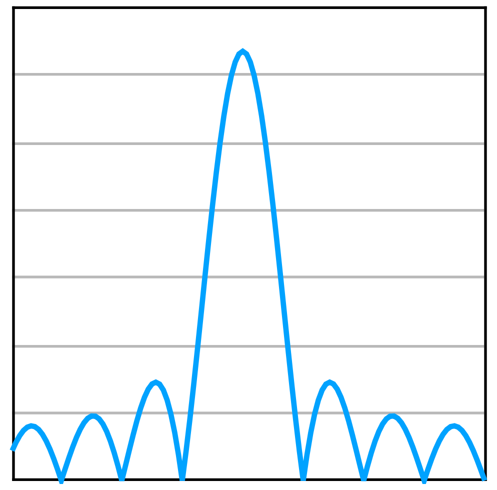
Fourier Transform of Standard Functions

Separability $$e^{j(\omega_x x+\omega_y y)} = e^{j\omega_x x} \cdot e^{j\omega_y y}$$

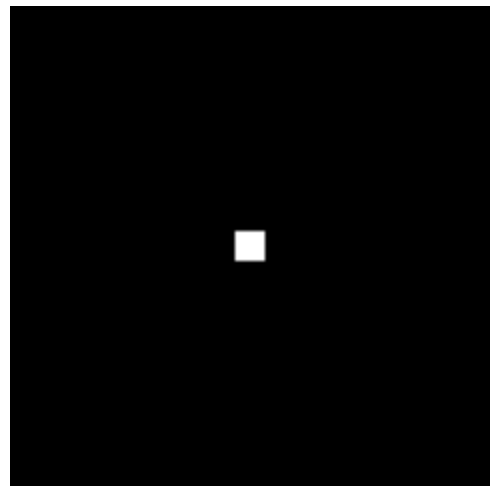
$$f(x,y) = f_1(x) \cdot f_2(y) \iff \hat{f}(\omega_x,\omega_y) = \hat{f}_1(\omega_x) \cdot \hat{f}_2(\omega_y)$$



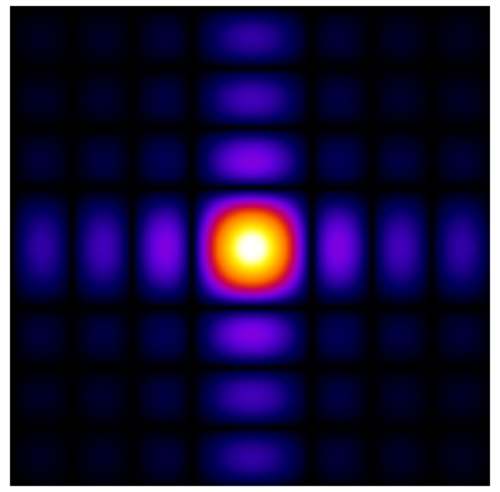
Signal 1D



Fourier 1D spectrum



Signal 2D



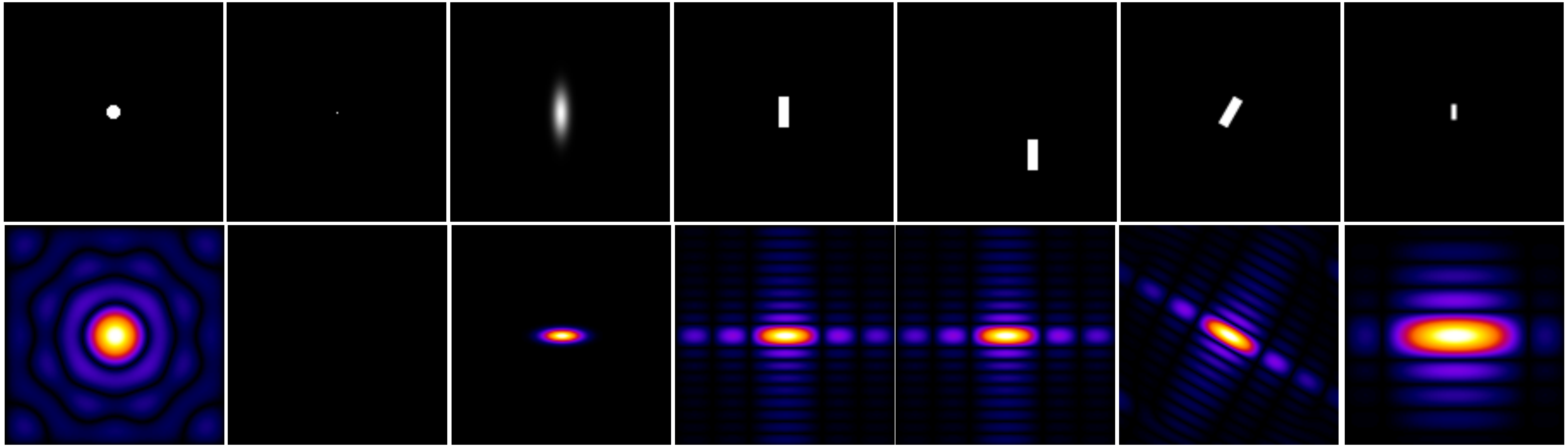
Fourier 2D spectrum

Typical Fourier Transforms and Properties

Radial frequency	Impulse ➠ constant	Gaussian ➠ Gaussian	rect ➠ sinc	translate ➠ invariant	rotation ➠ rotation	reduce ➠ enlarge

Spatial 2D

Fourier 2D



Convolution

Most important operation in image-processing

- Model for the image acquisition
- Filtering
- Deconvolution

Continuous convolution $$(g * h)(\boldsymbol{x}) = \int_{\mathbb{R}^d} h(\boldsymbol{y})\, g(\boldsymbol{x} - \boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$$

2D discrete convolution $$(g * h)(k, l) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} h[m, n]\, g[k - m, n - l]$$

Properties

$$f * g = g * f$$

$$f * (g * h) = (f * g) * h$$

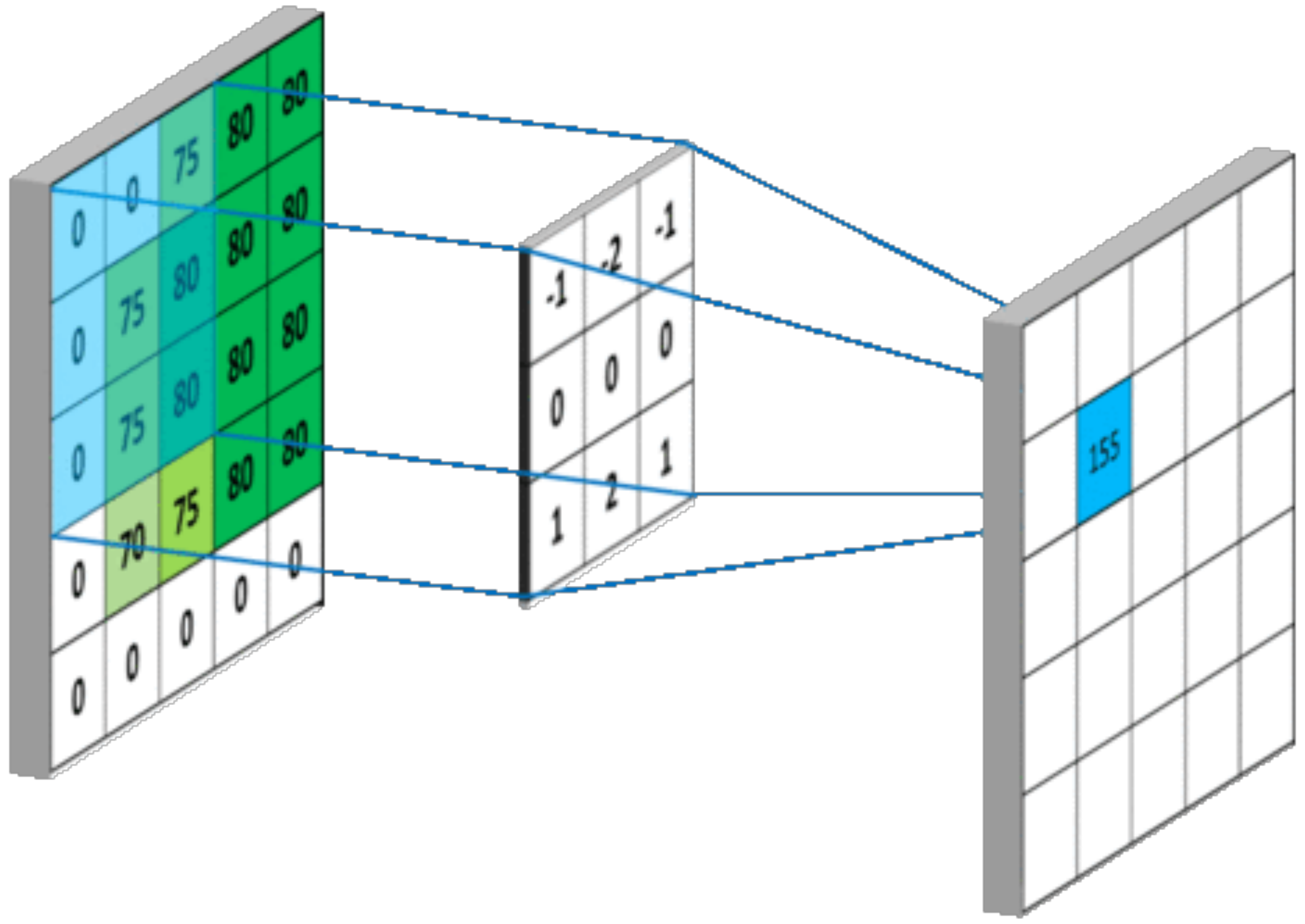
$$f * (g + h) = (f * g) + (f * h)$$

$$f * \delta = f$$

$$f^{(-1)} * f = \delta$$

$$(f * g)' = f' * g = f * g'$$

- h has generally a small support (M)
- h is the impulse response of a filter
- h is the mask (a.k.a kernel) of the filter



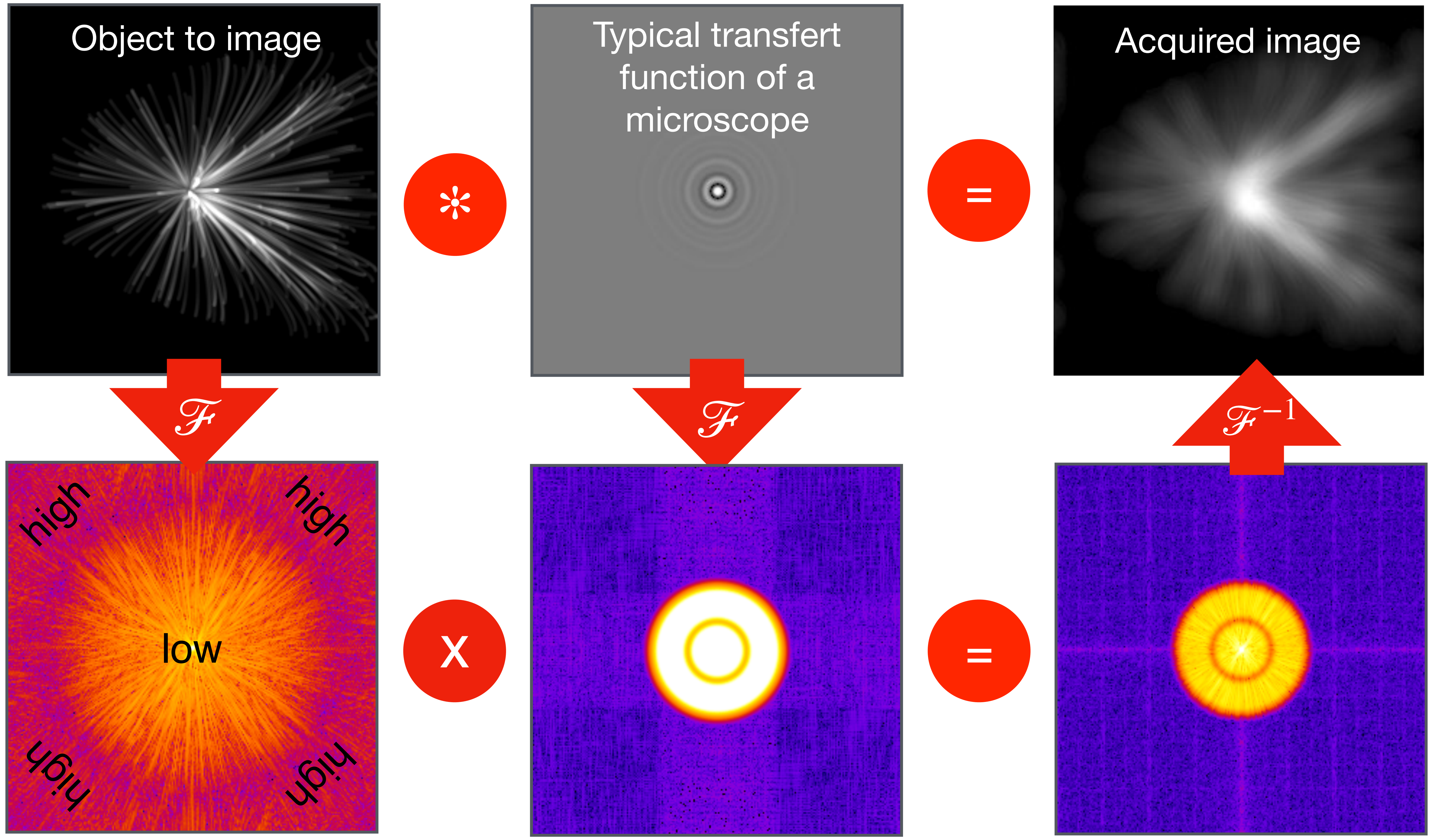
Theorem of Convolution

Circular **convolution** in space is a point-wise **multiplication** in the Fourier domain

$$\mathscr{F}\{f(\boldsymbol{x}) * g(\boldsymbol{x})\} = \mathscr{F}\{f(\boldsymbol{x})\} \cdot \mathscr{F}\{g(\boldsymbol{x})\}$$

$$\mathscr{F}\{f(\boldsymbol{x}) \cdot g(\boldsymbol{x})\} = \mathscr{F}\{f(\boldsymbol{x})\} * \mathscr{F}\{g(\boldsymbol{x})\} \cdot \frac{1}{(2\pi)^d}$$

Simulation of the image formation



Number of floating operations for a FFT (N length of signal) :

$N \log N$

$(N_x N_y N_z)^2$

64x64x64: 2 min • 0.25 Mb

512x512x256: 100 days • 64 Mb

$$f[m] = \sum_{m=0}^{N} h[m]\, g[k-m]$$

$2(N_x N_y N_z)\ \log(N_x N_y N_z) + (N_x N_y N_z)$

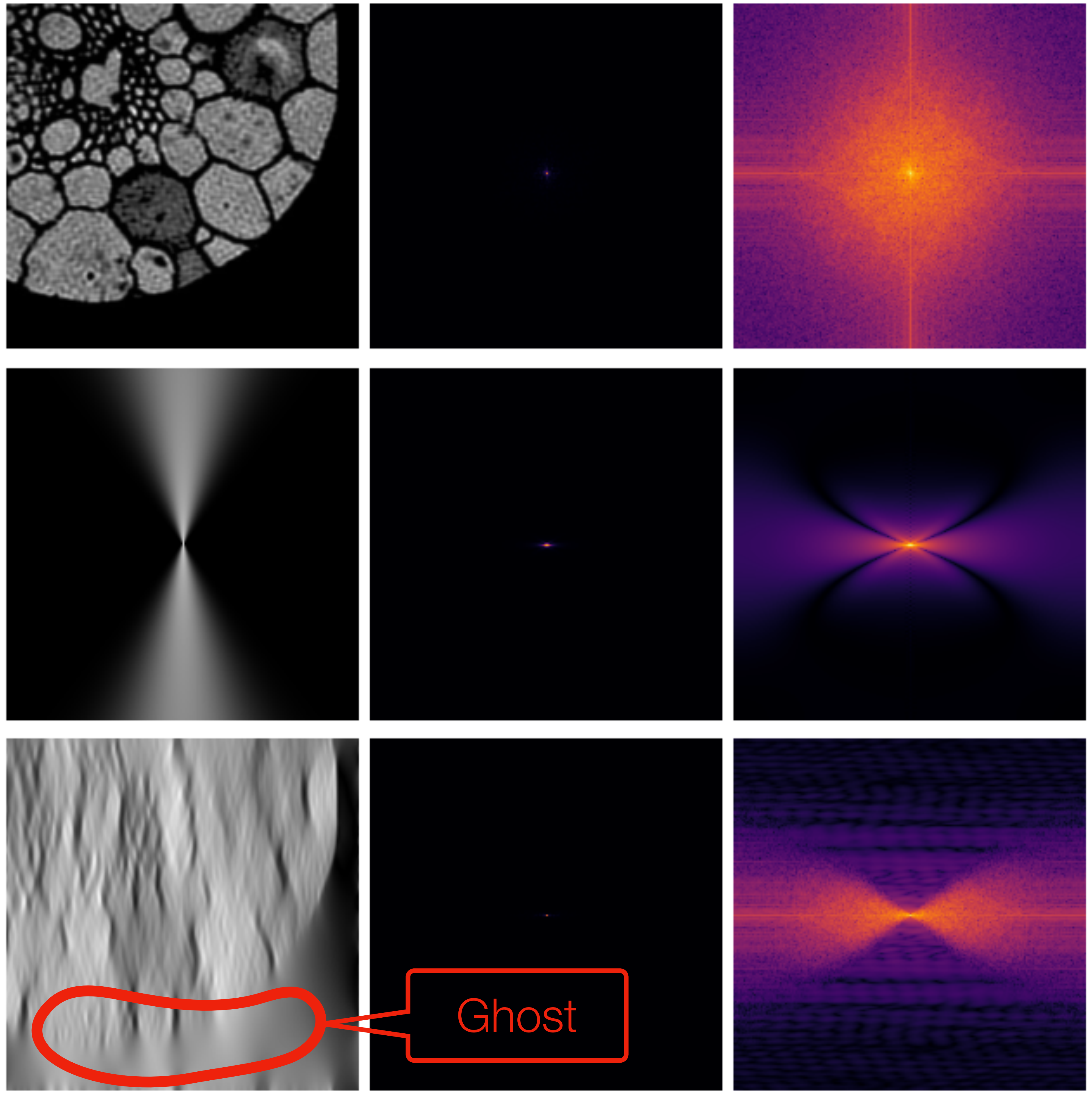
64x64x64: ~10 ms • 2 Mb

512x512x256: ~4 s • 512Mb

$$\hat{f}[\omega] = \hat{h}[\omega] \cdot \hat{g}[\omega]$$

Convolution in the Fourier Domain

Simulation of a pinhole
lateral plane



space | Fourier | Fourier (log)

Circular convolution

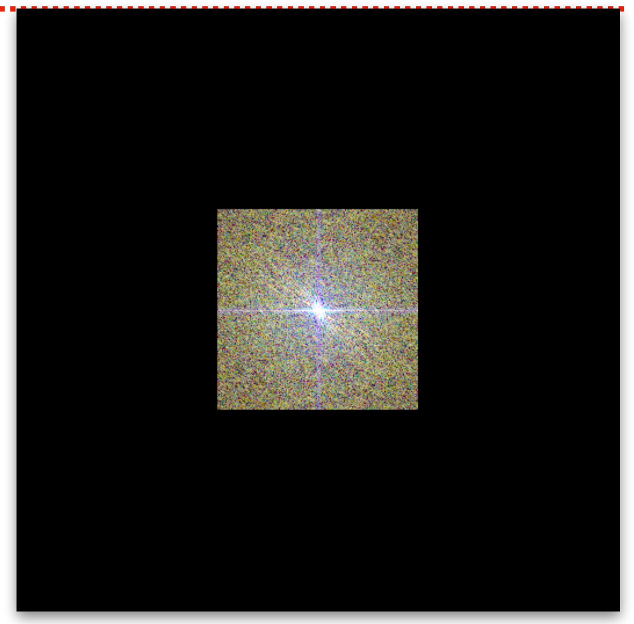
Periodic basis function

➔ **Artefact at the border**

Padding

Extend by zero-padding

➔ **more memory/time**

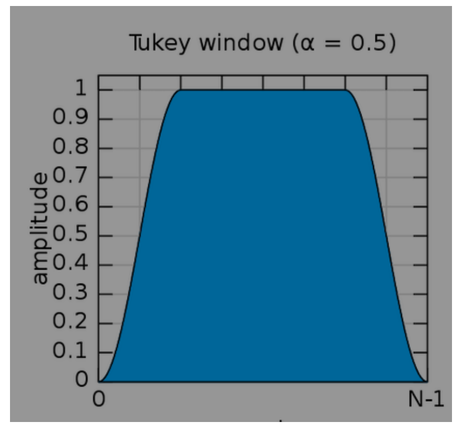


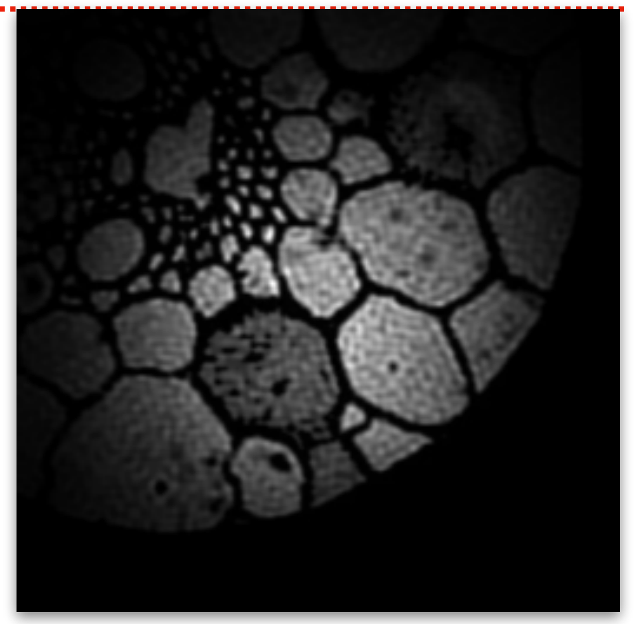
Apodization

Set the edges to 0 by a windowing function

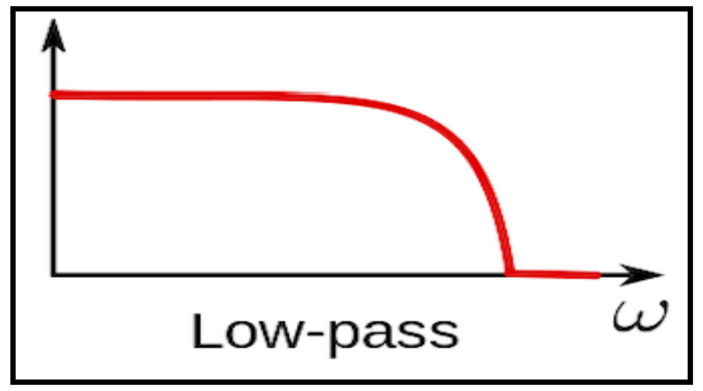
➔ **Energy loss**

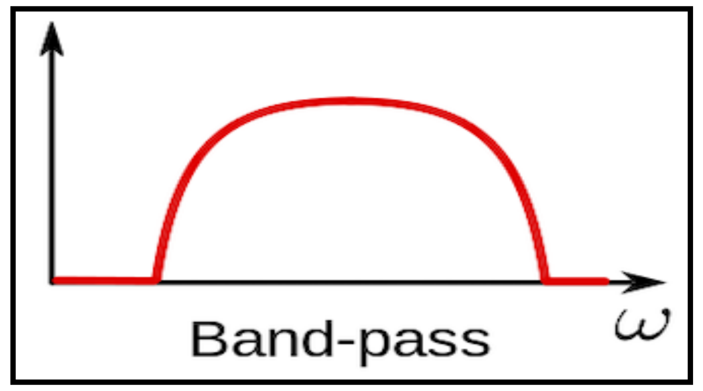
Turkey, Hann,
Cosine, Hamming

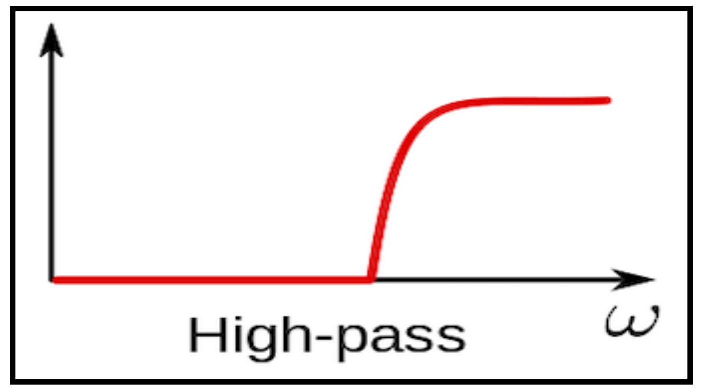


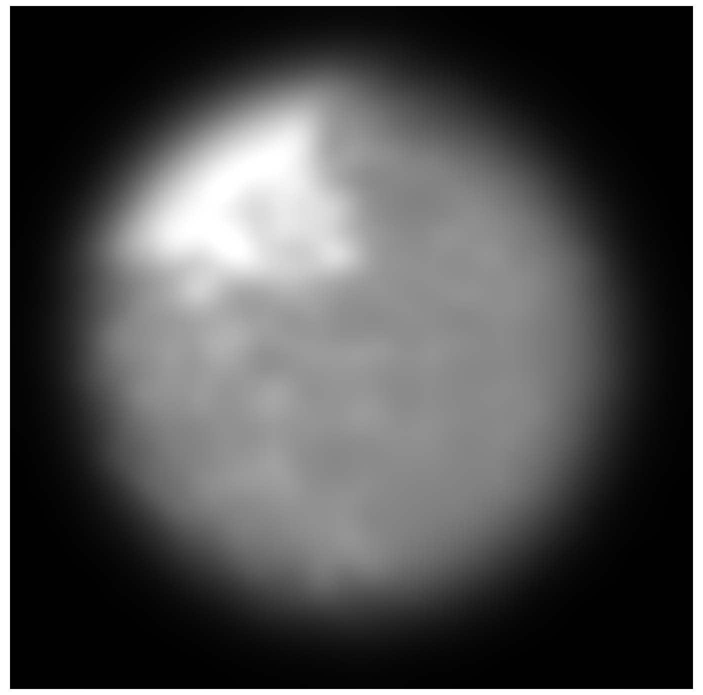


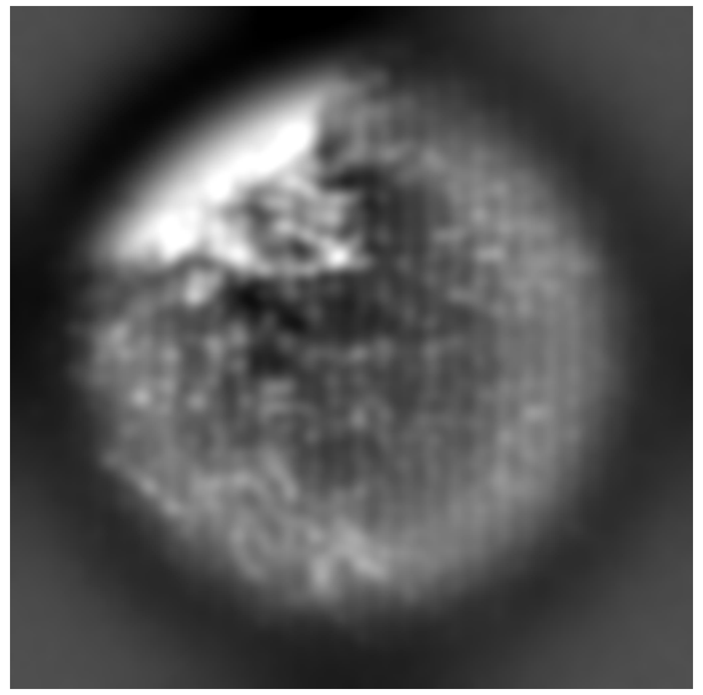
Filtering in the Fourier Domain

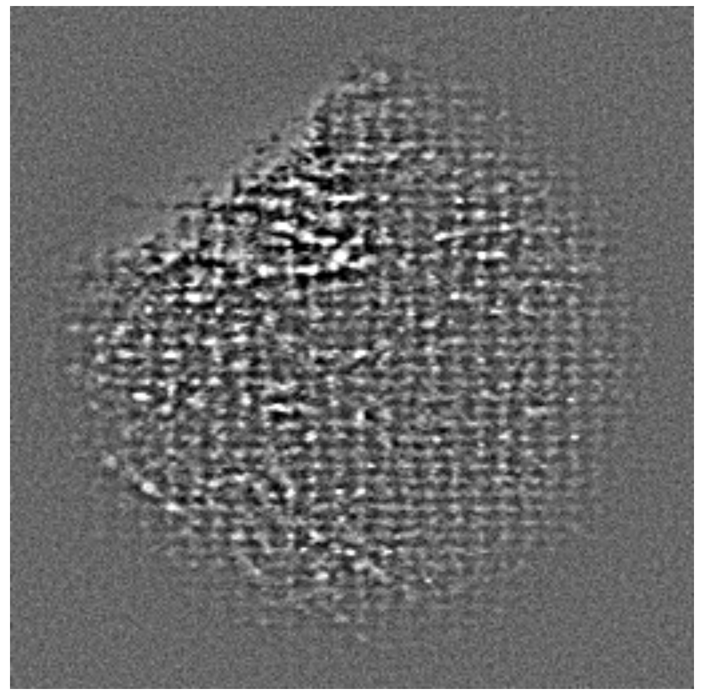


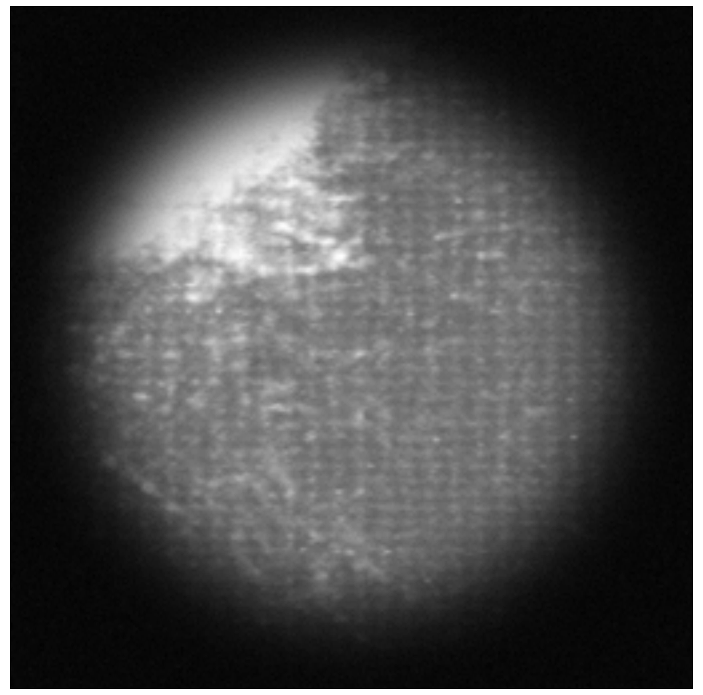


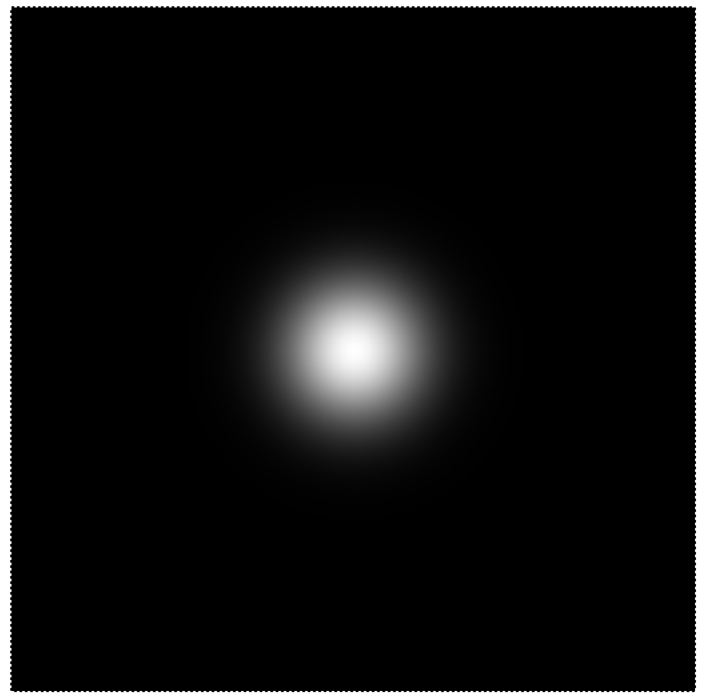


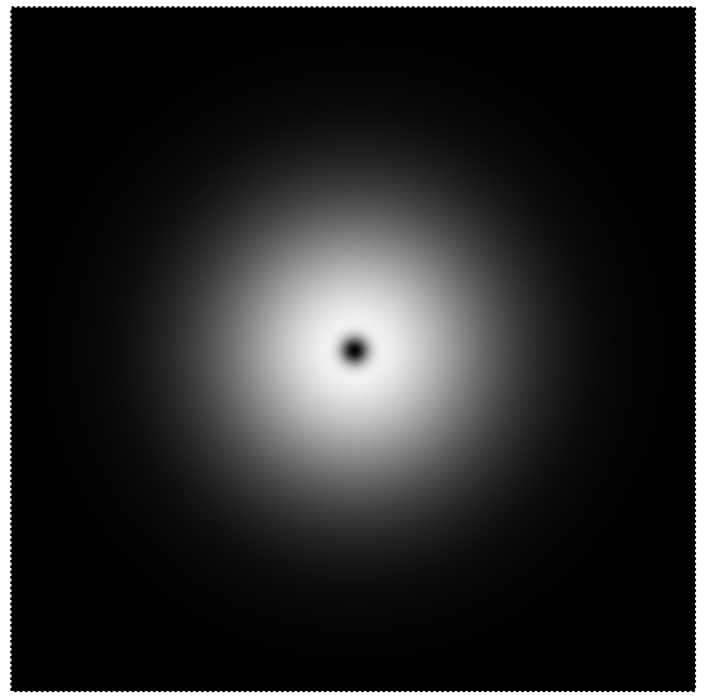


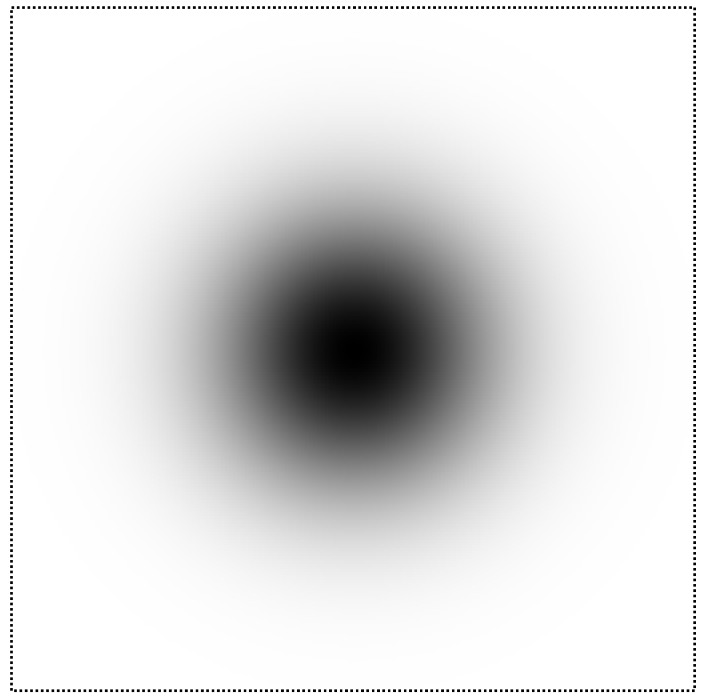












Demonstration

Fourier Filtering

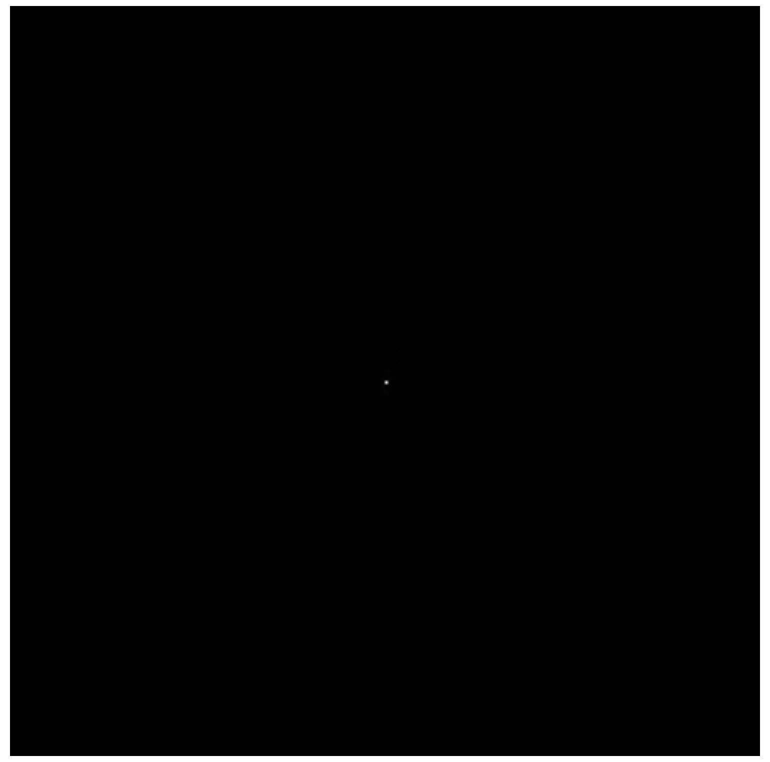
http://bigwww.epfl.ch/demo/ip/demos/FFT-filtering/

Jupyter Notebook

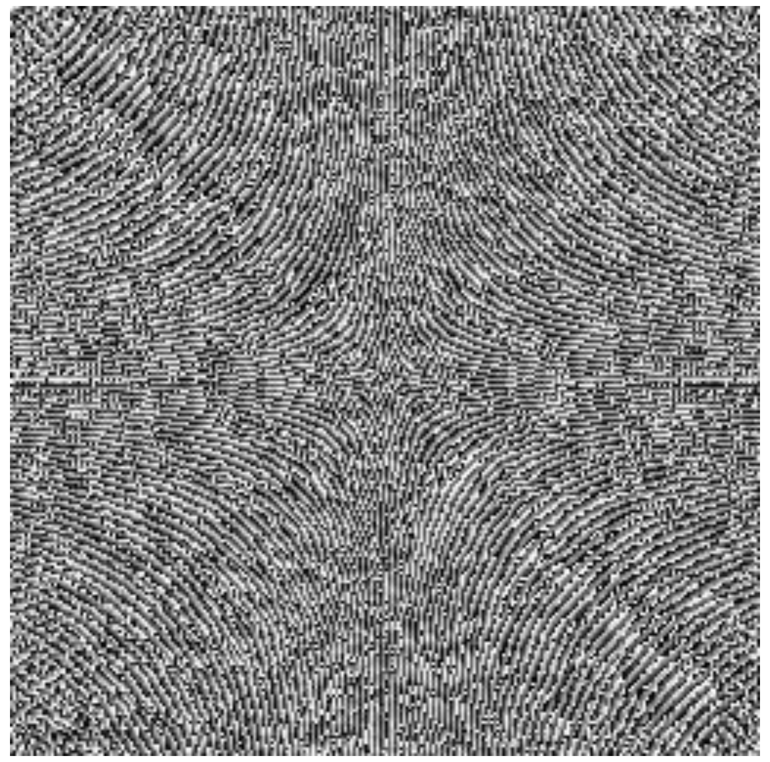
Phase Information

a

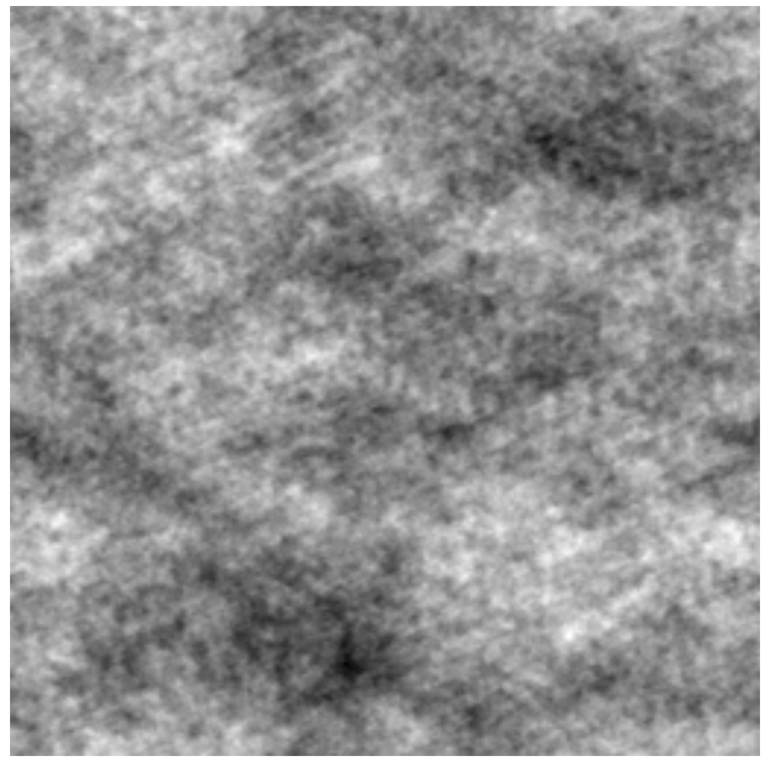
modulus of a



phase of a

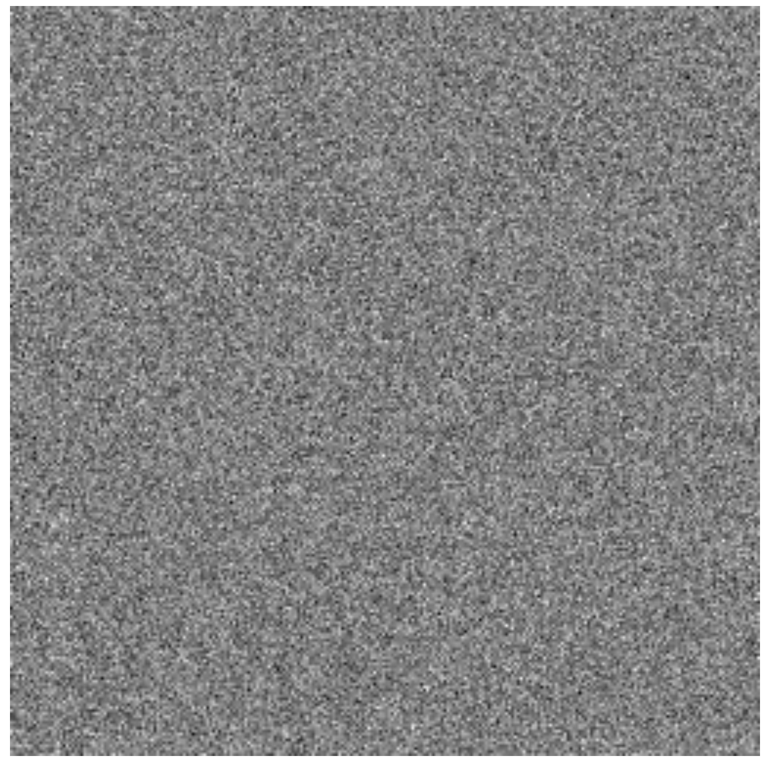


modulus of a and phase of b

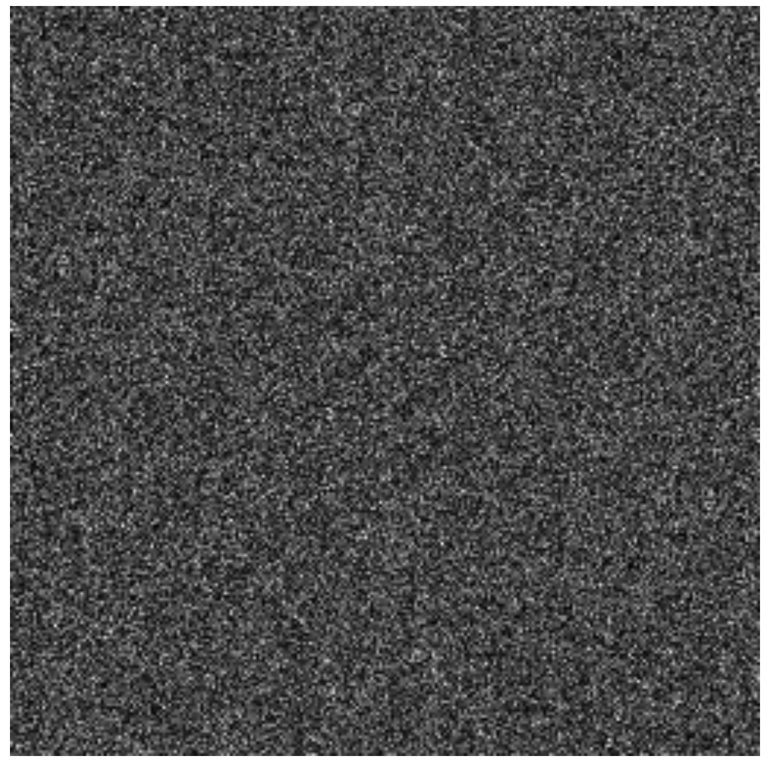


Modulus encodes amplitude

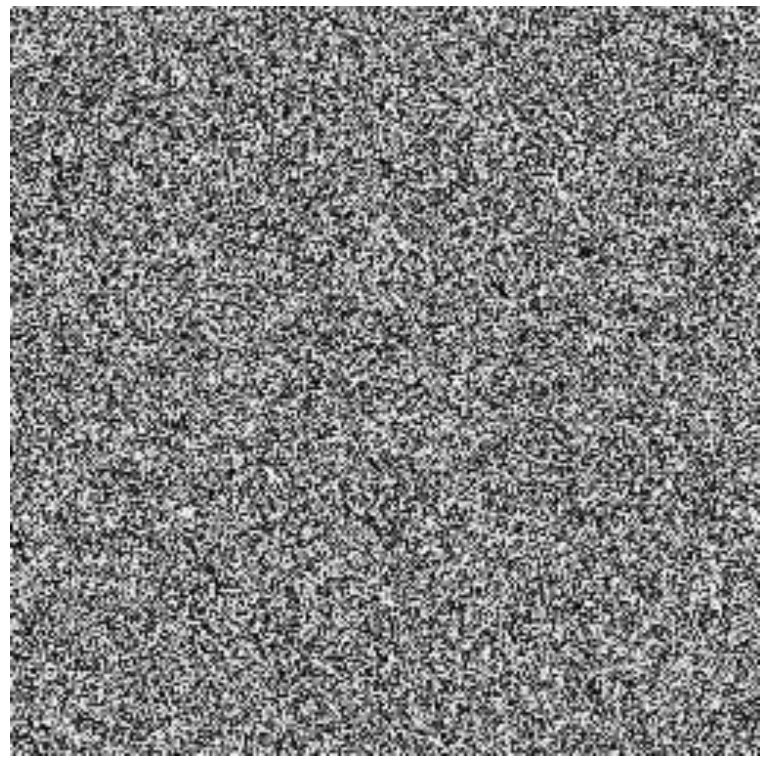
b



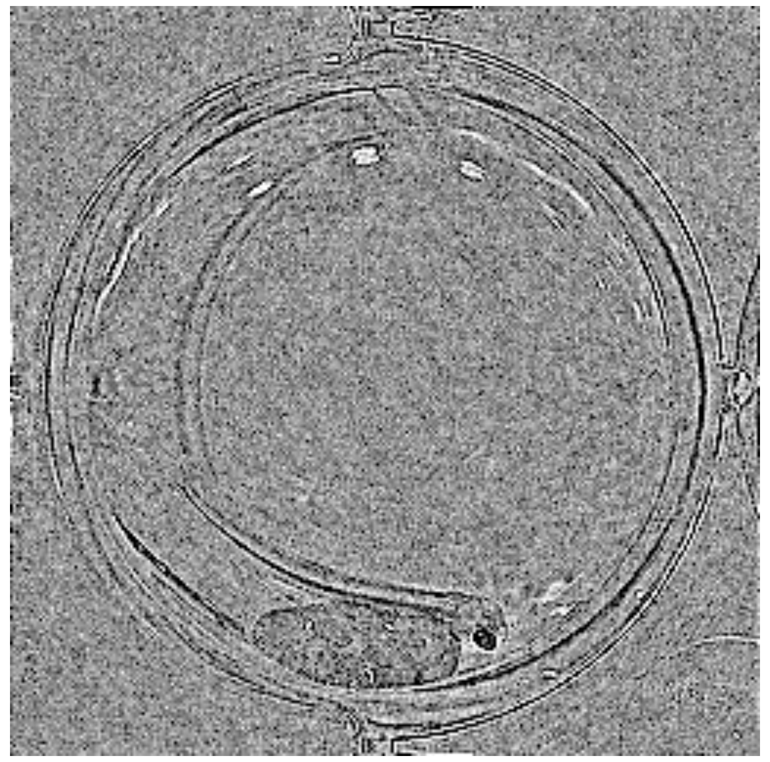
modulus of b



phase of b



modulus of b and phase of a

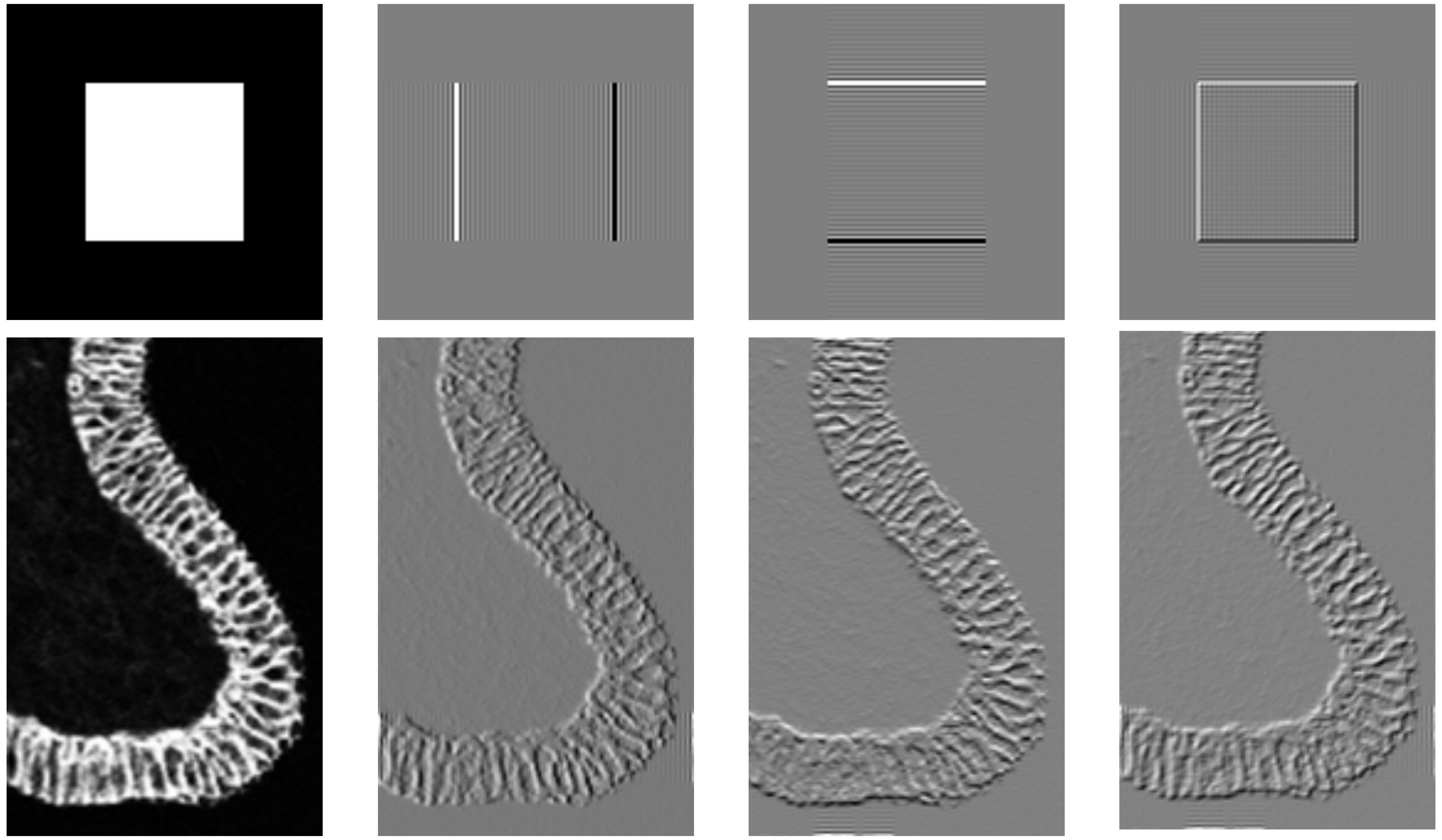


Phase encodes position of edges

Edges are important for visual perception

Differentiation

$$\frac{\partial f(\boldsymbol{x})}{\partial x_i} \overset{\mathscr{F}}{\longleftrightarrow} j\,\omega_i\,\hat{f}(\boldsymbol{\omega})$$

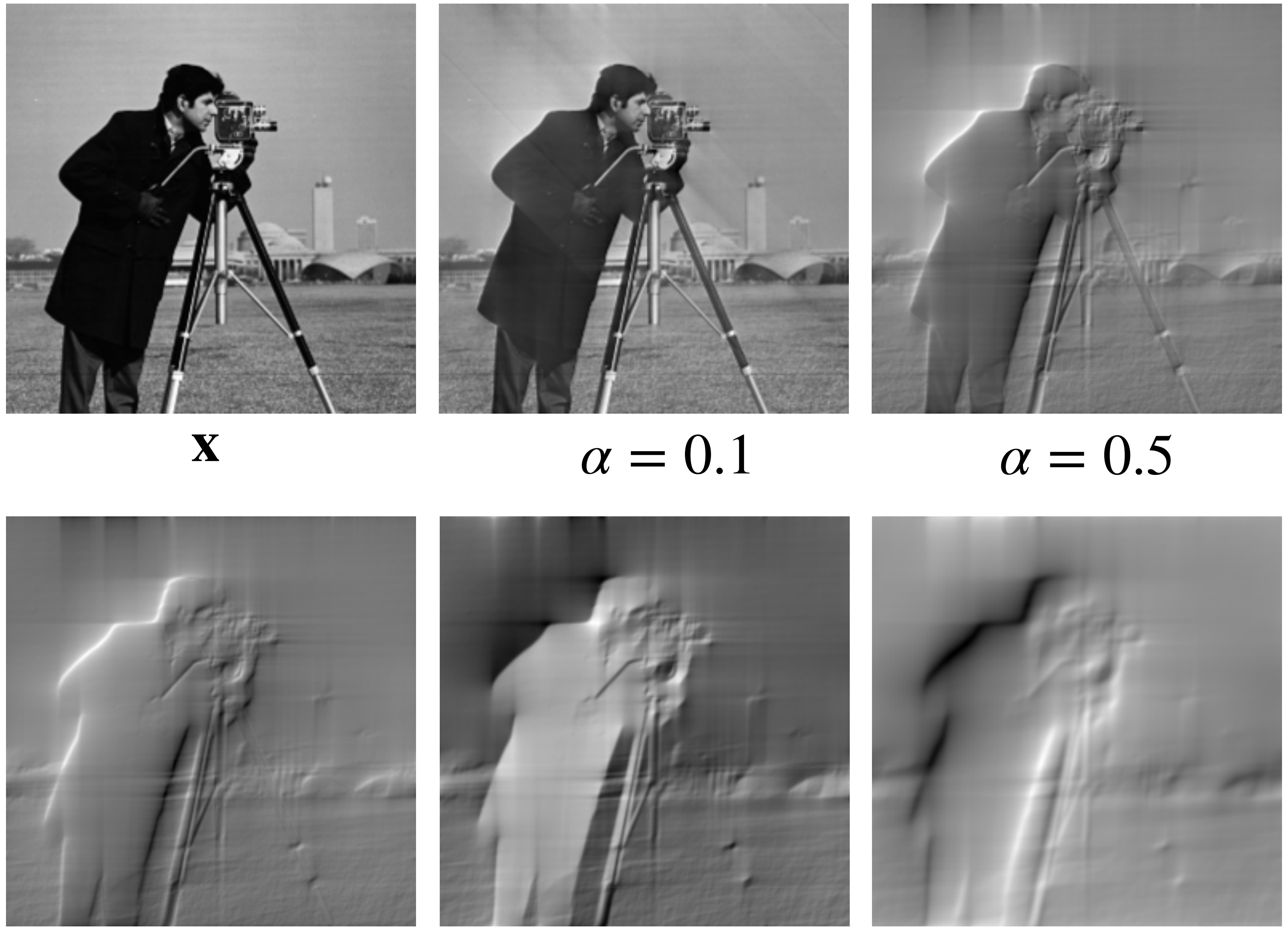


$j\omega_x$ $j\omega_y$ $j|\boldsymbol{\omega}|$

$$f'(x) = \mathscr{F}^{-1}\left(j\omega\mathscr{F}(f(x))\right)$$

type of filter?

$$\frac{\partial^\alpha f(\boldsymbol{x})}{\partial x_i^\alpha} \overset{\mathscr{F}}{\longleftrightarrow} (j\omega_i)^\alpha\,\hat{f}(\boldsymbol{\omega})$$



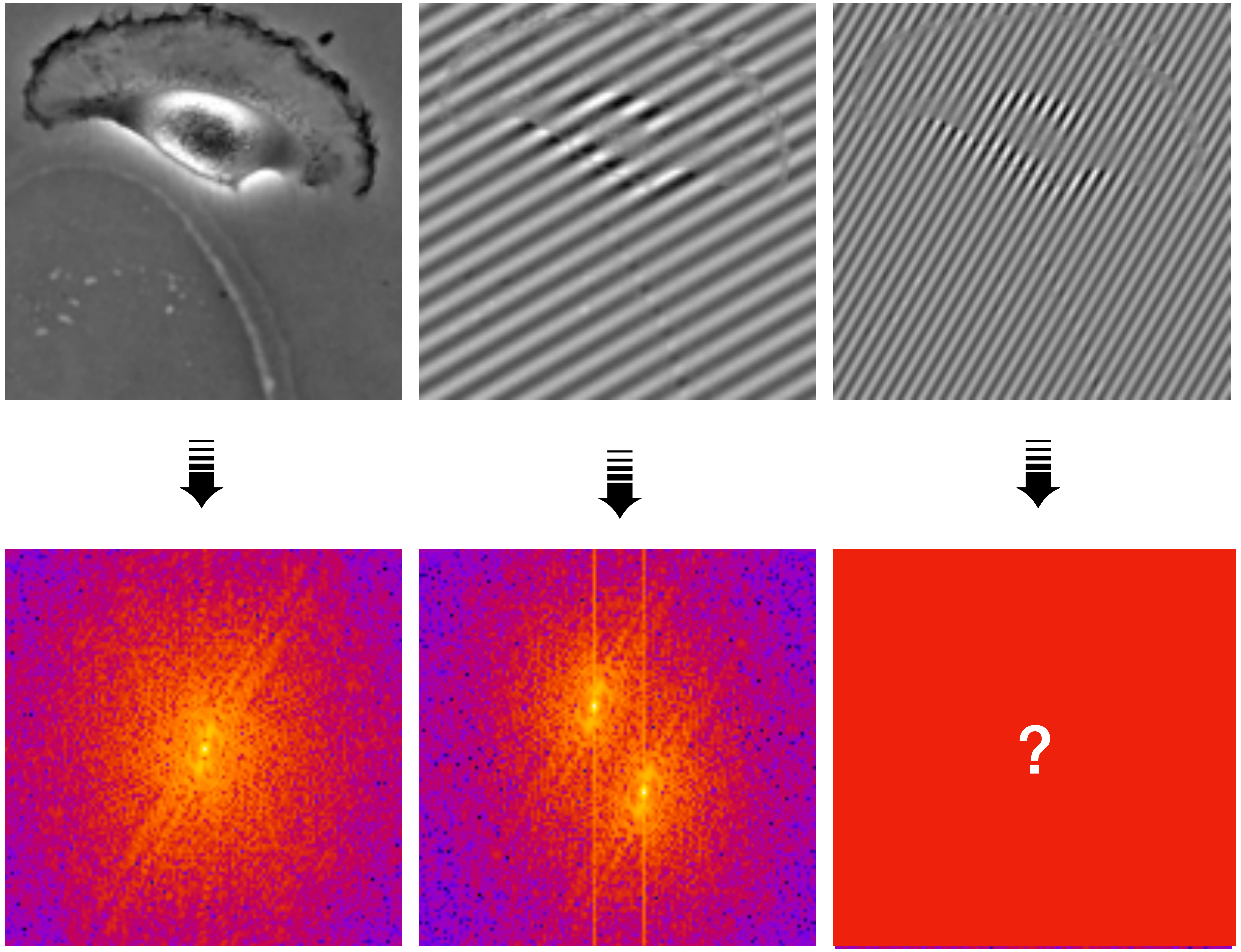
$\mathbf{x}$ $\alpha = 0.1$ $\alpha = 0.5$

$\alpha = 1$ $\alpha = 2$ $\alpha = 7$

Fractional derivatives

Modulation

$$e^{j\langle \boldsymbol{\omega}_0, \boldsymbol{x}\rangle} f(\boldsymbol{x}) \quad \xleftrightarrow{\mathscr{F}} \quad \hat{f}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$$



👉 Interference, Moiré pattern

Structured illumination

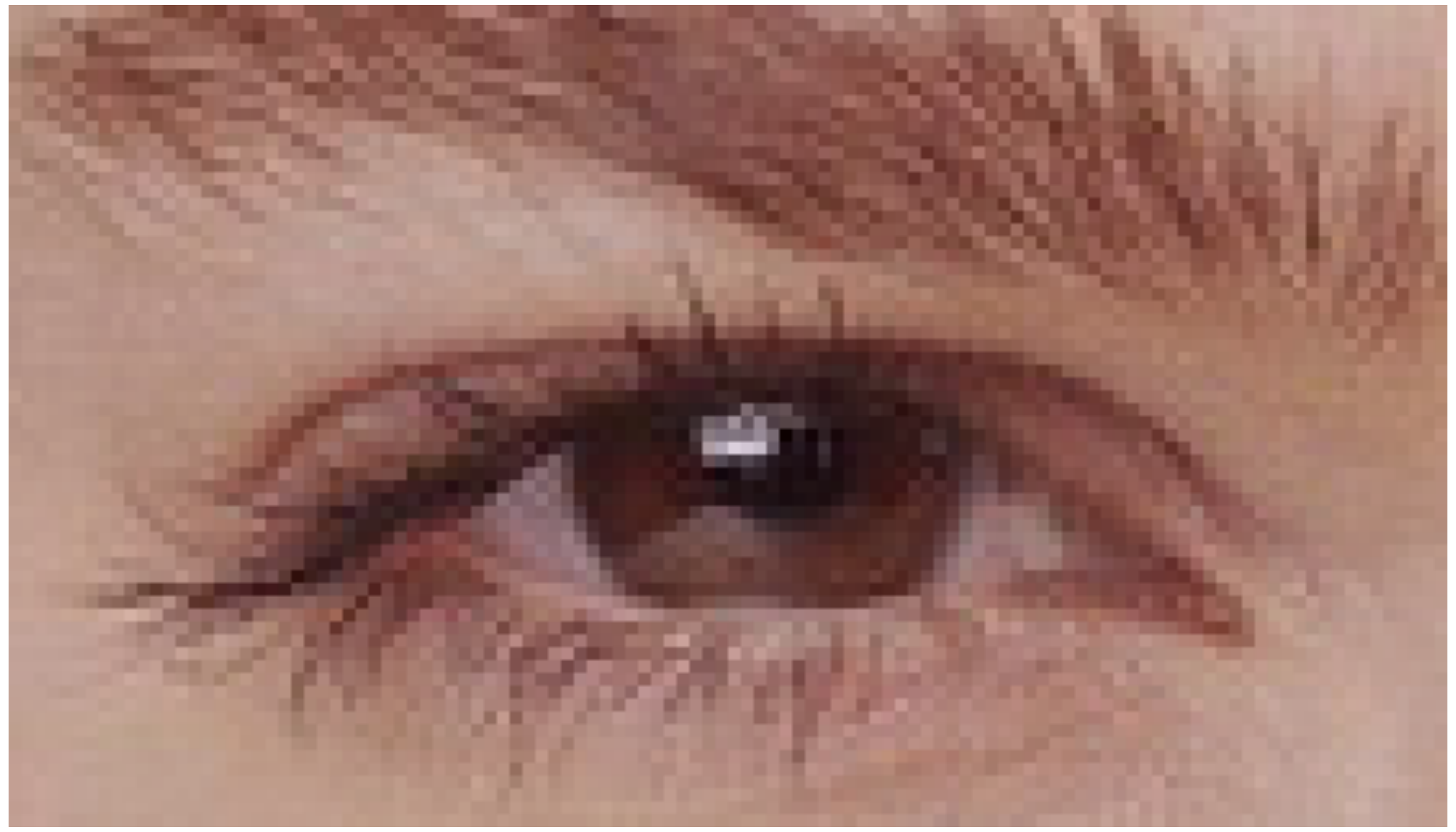
👉 **Super-resolution Microscopy**

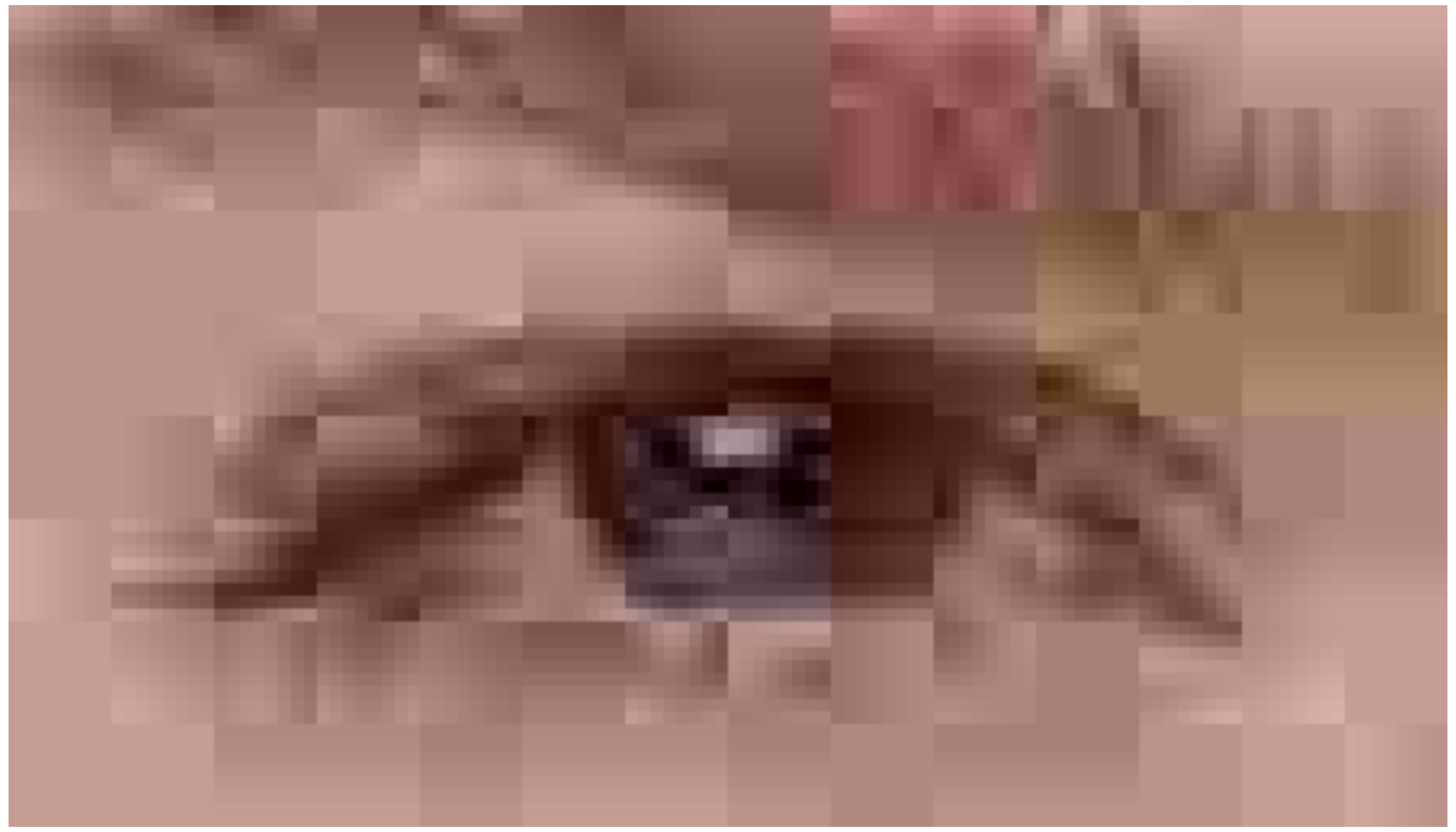
Structure Illumination Microscopy

Application Signal Compression

Using the Fourier Transform

- Keep higher coefficients
- MP3, JPEG, MPEG

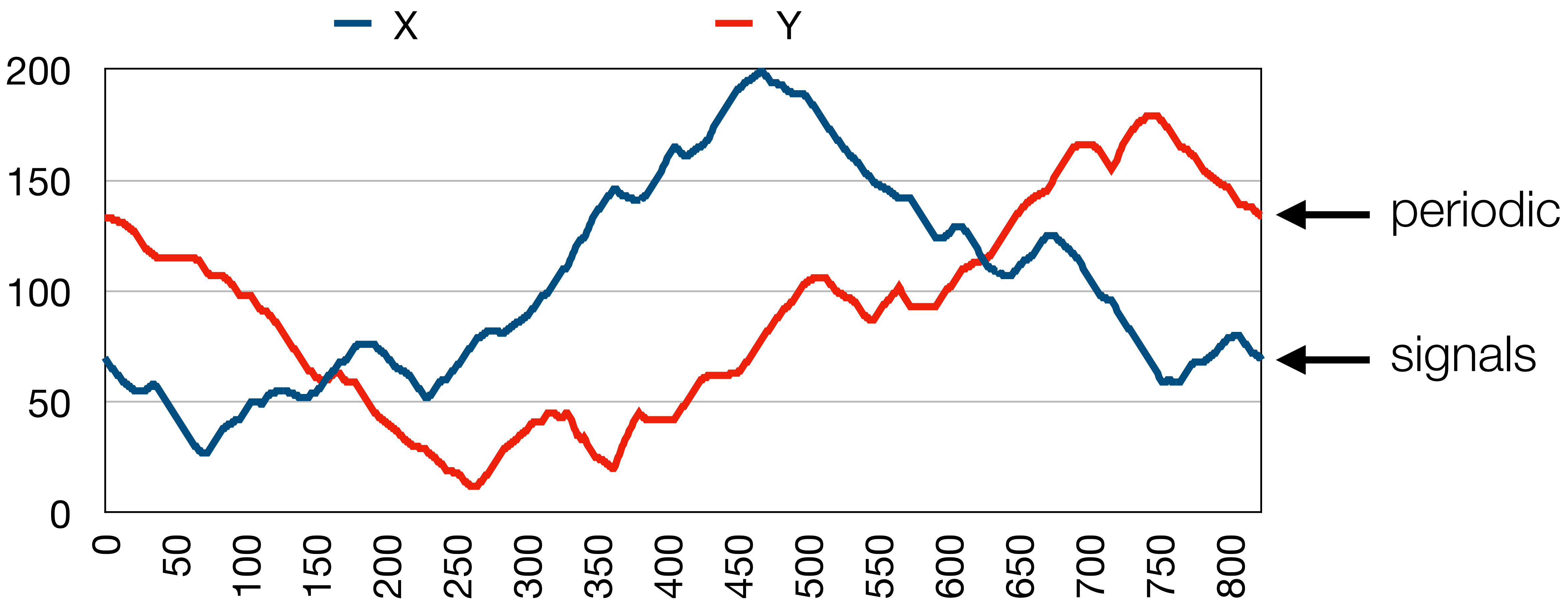


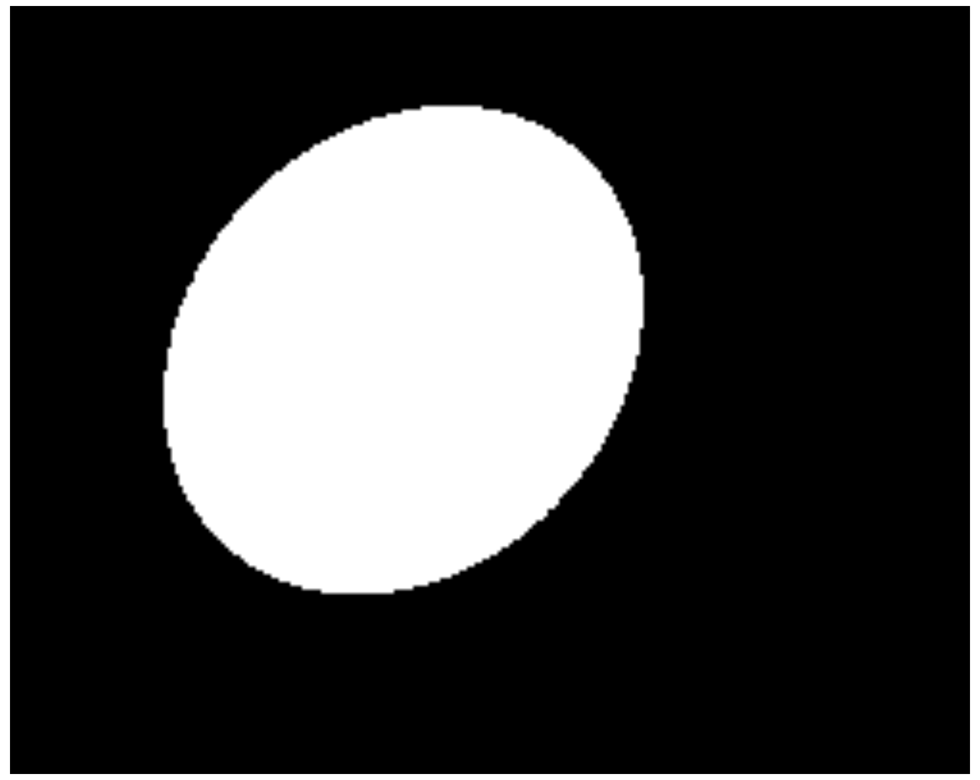


JPEG
DCT (Discrete Coefficients Transform) in 8x8 block

Application Fourier Descriptors

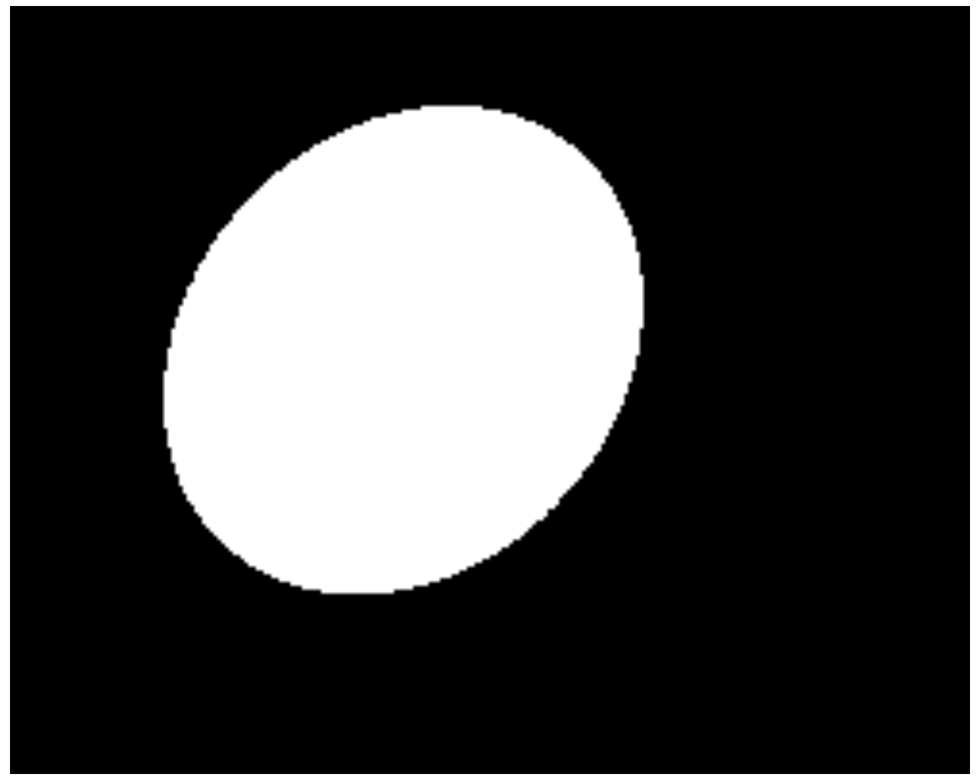
source





2 coefficients

8 coefficients



RECAP

- MULTIDIMENSIONAL FOURIER TRANSFORM
- COMPLEX REPRESENTATION, MODULE, PHASE
- INTERPRETATION OF FREQUENCIES
- PERFECT / PARTIAL RECONSTRUCTION
- READING THE FREQUENCY DOMAIN
- FILTERING, TYPE OF FILTER
- DIFFERENTIATION
- MODULATION, GET HIGHER RESOLUTION