Exercise 6: Population dynamics model with a limit cycle

October 24, 2023

Course: BIO-341 *Dynamical systems in biology*

Professor: *Julian Shillcock & Felix Naef*

SSV, BA5, 2023

Note that this document is primarily aimed at being consulted as a Jupyter notebook, the PDF rendering being not optimal.

[3]:
```python
#import important libraries
import numpy as np
import matplotlib.pyplot as plt
from ipywidgets import interact
from scipy.integrate import odeint
from IPython.display import set_matplotlib_formats
from matplotlib.markers import MarkerStyle
set_matplotlib_formats('png', 'pdf')
```

```
/var/folders/yf/q5hy0j9s5td1mjcljsx68fxh0000gq/T/ipykernel_24339/3074590393.py:8
: DeprecationWarning: `set_matplotlib_formats` is deprecated since IPython 7.23,
directly use `matplotlib_inline.backend_inline.set_matplotlib_formats()`
  set_matplotlib_formats('png', 'pdf')
```

1 Population dynamics model with a limit cycle

In the course, we studied predator-prey models with a stable spiral, which means that the two populations settle to coexist after the oscillatory transient decay.

Here, we study the following model:

$$\frac{dN}{dt} = a\,N(K-N) - c\,P\frac{N}{N+R} \tag{1}$$
$$\frac{dP}{dt} = b\,P\,(sN-P) \tag{2}$$

where the predators P might represent C. Elegans worms feeding on E. Coli bacteria N (typically used as food in laboratory dishes). All parameters should be taken positive (> 0).

1) Explain the different parameters in the model (a,b, c, R, K, s). What are their units?

Hint: Make the connection with the logistic growth model.

Solution

a and b have units of $1/(\text{time} \cdot \text{number of individuals})$. aK represents the preys relative growth rate when N is small, and bsN represents the relative growth rate of the predators when P is small.

c has unit of 1/time, and it defines the rate of interaction between predators and preys.

K is the carrying capacity of the environment for the preys, its unit is number of individuals.

R 'protects' the preys from being eaten by the predators (it can represents an alternative food resource for the predators, for example), its unit is number of individuals.

s is adimensional and it is a scaling parameter.

2) Calculate and plot the nullclines for the following values of the parameters:

a = b = 0.01

c = 1.

K = 200

R = 50

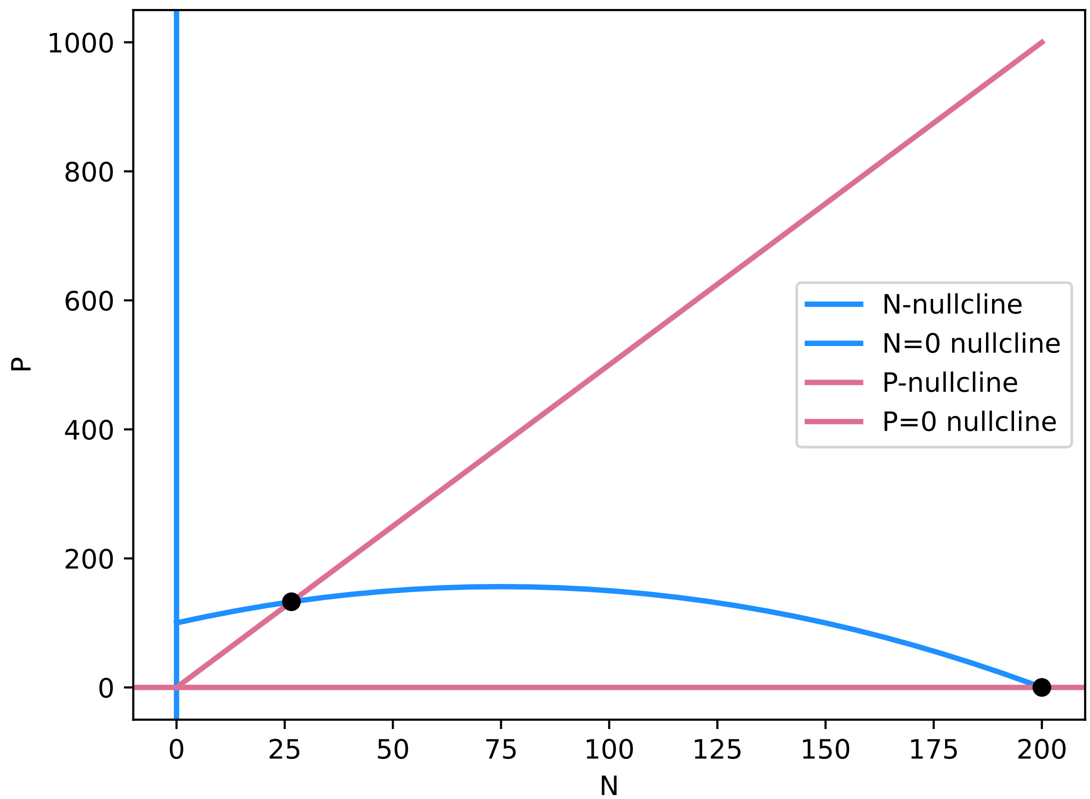
s = 5

Plot the fixed points and qualitatively describe their meanings in terms of populations of predators and preys

Solution

[4]:
```python
#nice color palette
import seaborn as sn
col = sn.color_palette("Blues",3)


N = np.linspace(0,200,500)

# paramters:
a = b = 0.01
c = 1
K = 200
R = 50
s = 5

plt.plot(N, a/c*(K-N)*(N+R), color='dodgerblue', label = 'N-nullcline', lw=2)
plt.axvline(x=0, color = 'dodgerblue', label = 'N=0 nullcline', lw=2)

plt.plot(N,s*N, color='palevioletred', label = 'P-nullcline', lw=2)
plt.axhline(y=0, color = 'palevioletred', label = 'P=0 nullcline', lw=2)
```

```
plt.plot(K,0, marker='o', color='k')

Q = c*s/a
N0 = ( -(R-K+Q) + np.sqrt((R-K+Q)**2 + 4*K*R) ) / 2
plt.plot(N0,s*N0,'ko')

plt.legend()
plt.xlabel('N')
plt.ylabel('P')
plt.show()
```



3) Plot the stability of the fixed point with both N,P $\neq$ 0 in the R-K plane (for K, R in [5, 500])

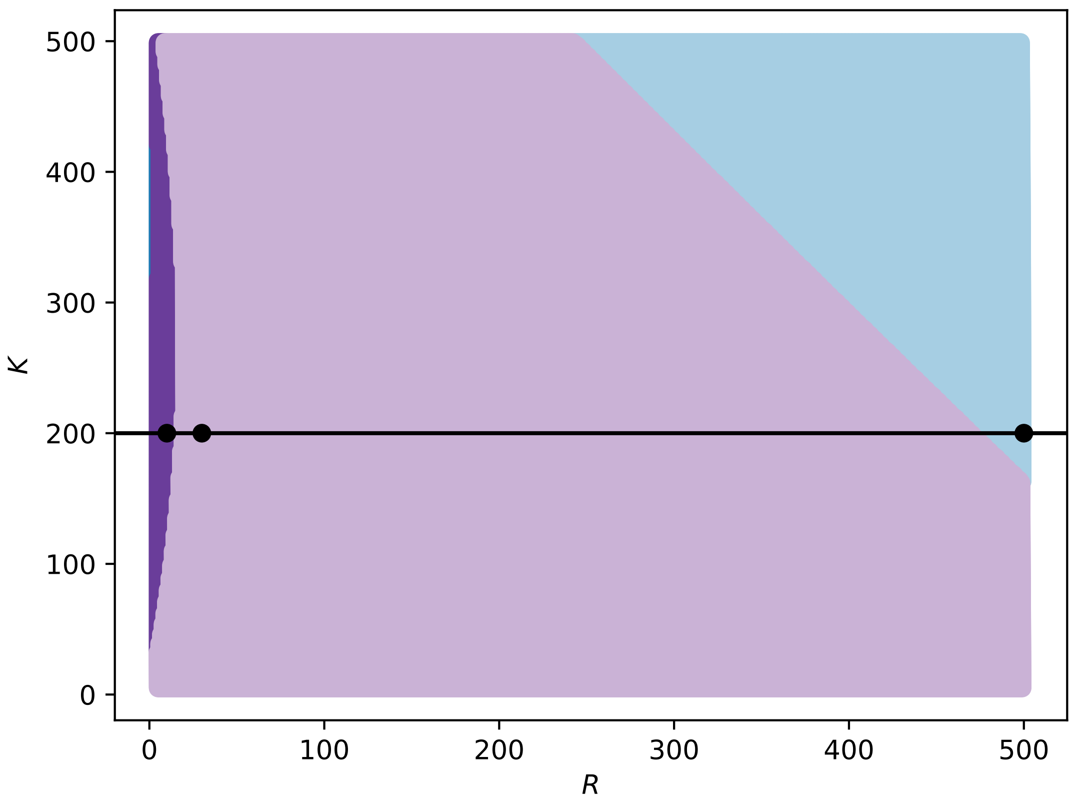
[5]:
```
def trace(M):
    return M[0,0]+M[1,1]

def determinant(M):
    return M[0,0]*M[1,1]-M[1,0]*M[0,1]
```

```
def stability(M):
    tau = trace(M)
    delta = determinant(M)
    if delta < 0:
        type_fp = 'saddle point'
    elif delta == 0:
        type_fp = 'non-isolated FP'
    else:
        if tau**2-4*delta < 0:
            type_fp_2 = 'spiral'
        elif tau**2-4*delta == 0:
            type_fp_2 = 'star'
        else:
            type_fp_2 = 'FP'

        if tau < 0:
            type_fp = 'stable ' + type_fp_2
        elif tau == 0:
            type_fp = 'center'
        else:
            type_fp = 'unstable ' + type_fp_2
    return type_fp
```

[6]:

```
from time import time
import seaborn as sns

palette = sns.color_palette("Paired")

colorandum = {'saddle point':palette[4], 'unstable FP':palette[1], 'stable FP':
↪palette[0], 'unstable spiral':palette[9], 'stable spiral':palette[8], 'center':
↪'k'}

def matrix(R,K):
    Q = c*s/a
    N = ( -(R-K+Q) + np.sqrt((R-K+Q)**2 + 4*K*R) ) / 2
    P=s*N
    return np.array([[a*(K-2*N) - c*P/(N+R) * (1-N/(N+R)),
                      -c*N/(N+R)],
                     [b*s*P,
                      b*(s*N-2*P)]])

start = time()

# First, store the fixed points
dic_result = {'center':[], 'saddle point':[], 'unstable FP':[], 'stable FP':[],␣
↪'unstable spiral':[], 'stable spiral':[]}
```

```
K = np.arange(5,500,1)
R = np.arange(5,500,1)

for k in K:
    for r in R:
        M = matrix(r,k)
        type_fp = stability(M)
        dic_result[type_fp].append([r,k])
# Then, plot

for type_fp, coordinates in dic_result.items():
    # convert the list of coordinates into a 2d-array
    coordinates = np.array(coordinates)
    if coordinates.size!=0:
        # make a scatterplot for all coordinates of a given type of fp
        plt.scatter(coordinates[:,0], coordinates[:,1], label = type_fp,␣
 ↪color=colorandum[type_fp])

plt.xlabel(r'$R$')
plt.ylabel(r'$K$')
plt.axhline(200, c = 'k')
plt.scatter(x=np.array([10,30,500]), y=np.array([200,200,200]), c = 'k')


plt.show()


end = time()

print("Total time: {:.1f}s".format(end-start))
```




```
Total time: 7.3s
```

4) Fix K, and use the same values of the parameters for a,b,c and s as above. Choose 3 values of R for which you have respectively a stable f.p., a stable spiral and a limit cycle.

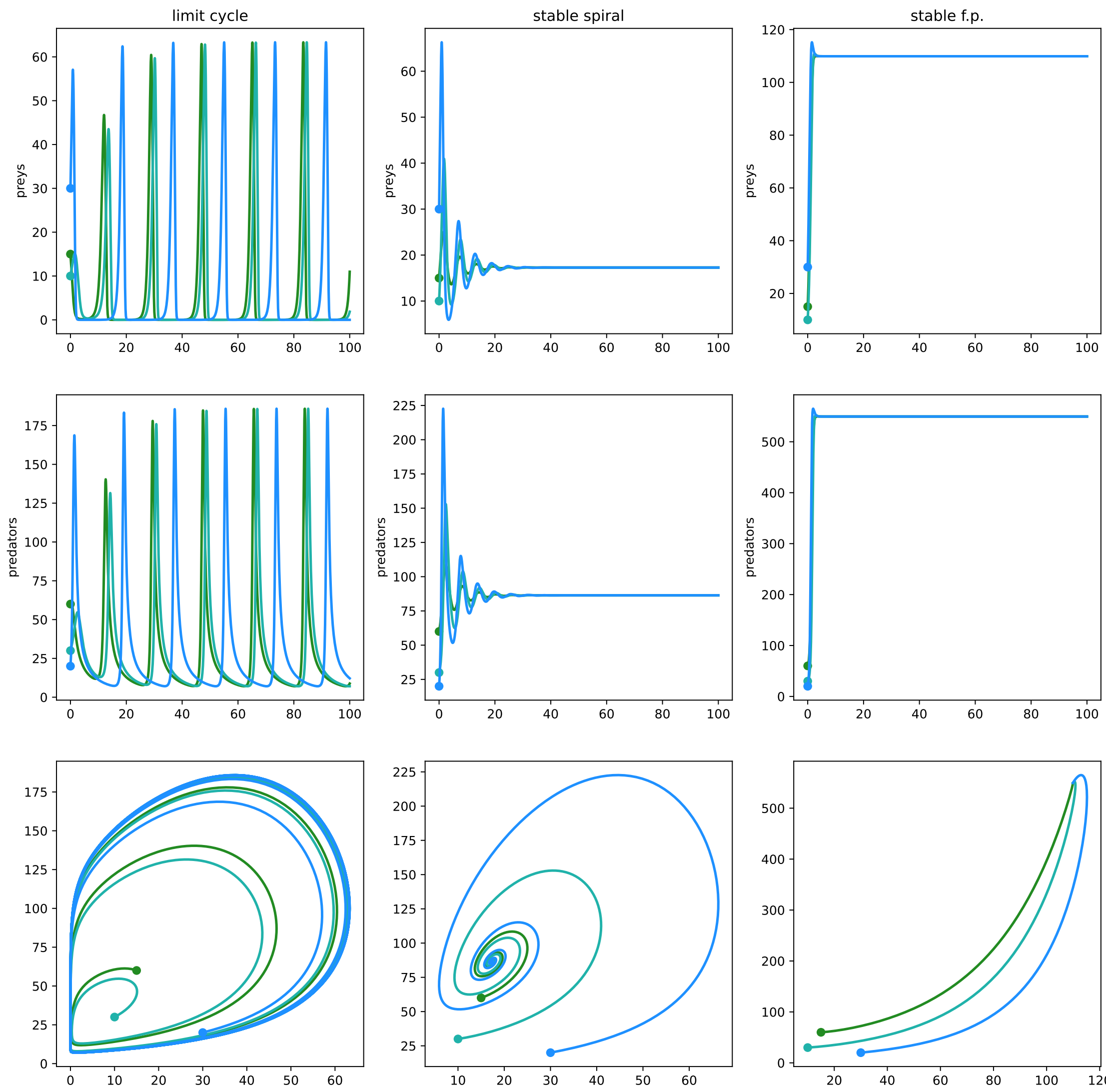
Simulate the trajectories (of N and P) using a python solver (ex: odeint). Plot the trajectories in function of time and the phase portrait (P in function of N) (use subplots) for the three cases with different initial conditions.

In the case of the limit cycle, what is the stability of the fixed point?

In the case of the limit cycle, the PF is unstable

[7]:
```
# paramters:
a = 0.01
b = 0.01
c = 1
K = 200
s = 5
tspan = np.linspace(0,100, 10000)
```

[8]:
```
#2D differential equation
def Xdot(X, t):
    N = X[0]
    P = X[1]
    return np.array([a*N*(K-N)-c*P*N/(N+R), b*P*(s*N-P)])


f, axs = plt.subplots(3, 3, figsize=(15,15))
colormap=['forestgreen','lightseagreen','dodgerblue']

R_list = [10,30,500]
title_list = ["limit cycle", "stable spiral", "stable f.p."]

for j,R in enumerate(R_list):
    for n,X0 in enumerate([np.array([15,60]), np.array([10,30]), np.
 ↪array([30,20])]):
        x = odeint(Xdot, X0, tspan)

        axs[0,j].plot(tspan, x[:,0], c=colormap[n], lw=2)
        axs[1,j].plot(tspan, x[:,1], c=colormap[n], lw=2)
        axs[2,j].plot(x[:,0], x[:,1], c=colormap[n], lw=2)

        axs[0,j].plot(tspan[0],x[0,0], c=colormap[n], marker='o')
        axs[1,j].plot(tspan[0],x[0,1], c=colormap[n], marker='o')
        axs[2,j].plot(x[0,0], x[0,1], c=colormap[n], marker='o')

        axs[0,j].set_ylabel("preys")
        axs[1,j].set_ylabel("predators")

        axs[0,j].set_title(title_list[j])


plt.show()
plt.close()
```



5) Describe in words the behavior of the trajectories in terms of the number of predators and preys for each case.

In the case of stable FP, predators and preys rapidly converge to a stable coexistence, while in the case of a stable spiral, they first oscillate for a while before reaching the stable point.

In the case of a limit cycle the oscillations settle to constant amplitude and slope. The trajectory always reaches the same limit cycle independently of the starting point. First, the preys population N grows happily. But the more preys there are, the more food the predators have, and the population P increases. Then, the predators eat so much preys N that at a certain point, there are almost no more preys. Suddenly, the predators population shrinks and the cycle starts a new round with preys N growing happily.

6. (OPTIONAL): Check your answers using euler's method

Euler's method is the simplest way to solve a differential equation numerically. In order to approximate the solution of :

$$\dot{x} = F(x(t)) \, , \, x(t_0) = x_0$$

We can write one step of the method as :

$$x(t \, + \, dt) \simeq x(t) \, + \, dt \, F(x(t))$$

for a specific timestep size dt.

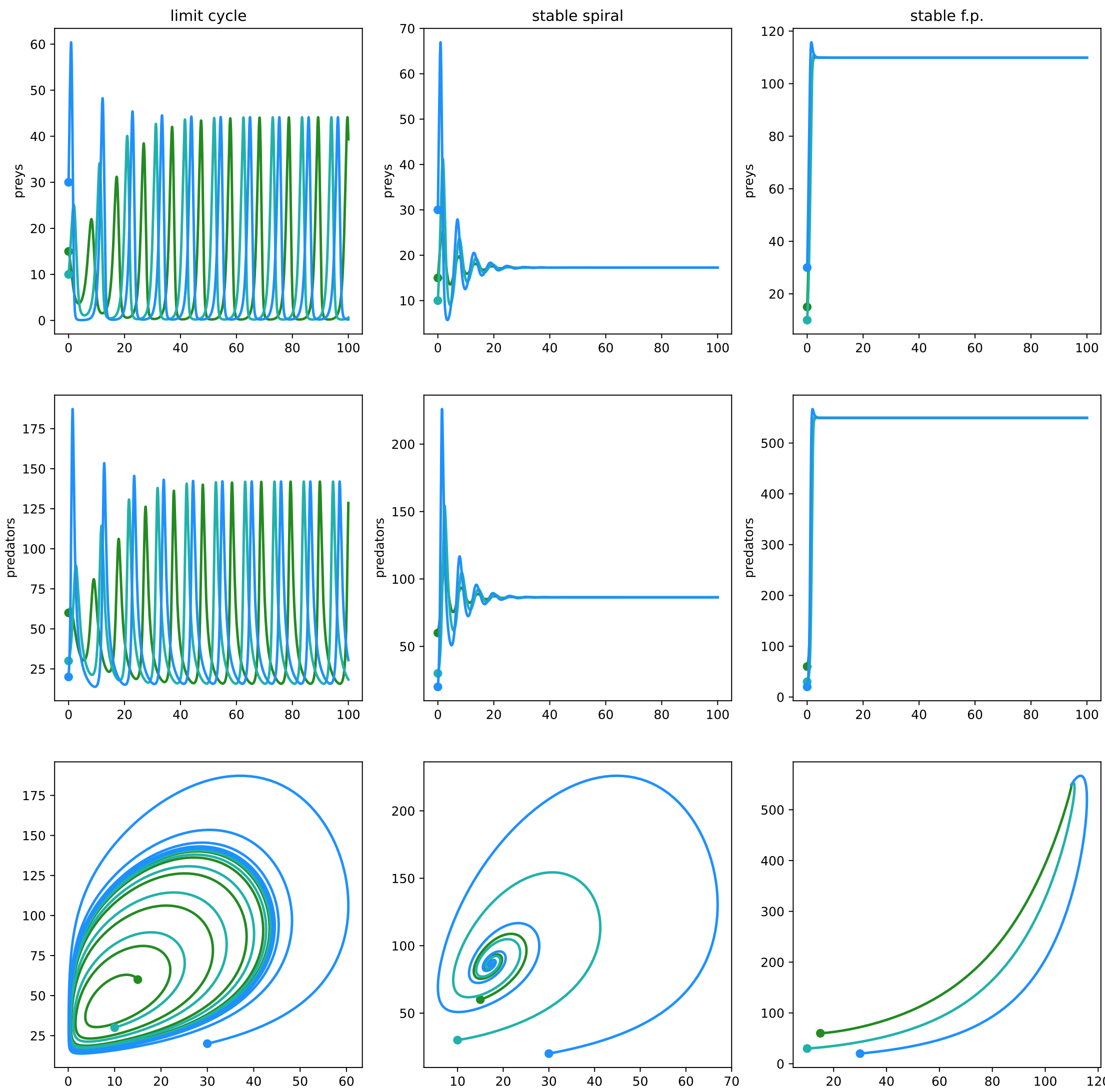
a) Implement your own Euler method using Python to solve numerically the following differential equation:

[9]:
```
#Euler's method
def Euler(xdot, x0, dt, T):
    x = x0
    t = 0
    l_x = [x0]
    l_t = [0]
    while t<T:
        x = x + xdot(x, t)*dt
        t+=dt
        l_x.append(x)
        l_t.append(t)
    return l_t, np.array(l_x)
```

b) Simulate the same trajectories than in 4) using the Euler's method.

[10]:
```
def Euler(xdot, x0, dt, T):
    x = x0
    t = 0
    l_x = [x0]
    l_t = [0]
    while t<T:
        x = x + xdot(x, t)*dt
        t+=dt
        l_x.append(x)
        l_t.append(t)
    return l_t, np.array(l_x) #array for l_x so it can easily be sliced for 2D␣
 ↪systems


#2D differential equation
def Xdot(X, t):
    N = X[0]
    P = X[1]
    return np.array([a*N*(K-N)-c*P*N/(N+R), b*P*(s*N-P)])
```

```python
# paramters:

a = 0.01
b = 0.01
c = 1
K = 200
s = 5

#simulation parameters
dt = 0.01
T = 100

f, axs = plt.subplots(3, 3, figsize=(15,15))
colormap=['forestgreen','lightseagreen','dodgerblue']

R_list = [15,30,500]
title_list = ["limit cycle", "stable spiral", "stable f.p."]

for j,R in enumerate(R_list):
    for n,X0 in enumerate([np.array([15,60]), np.array([10,30]), np.
 ↪array([30,20])]):
        t, x = Euler(Xdot, X0, dt, T)
        axs[0,j].plot(t, x[:,0], c=colormap[n], lw=2)
        axs[1,j].plot(t, x[:,1], c=colormap[n], lw=2)
        axs[2,j].plot(x[:,0], x[:,1], c=colormap[n], lw=2)

        axs[0,j].plot(t[0],x[0,0], c=colormap[n], marker='o')
        axs[1,j].plot(t[0],x[0,1], c=colormap[n], marker='o')
        axs[2,j].plot(x[0,0], x[0,1], c=colormap[n], marker='o')

        axs[0,j].set_ylabel("preys")
        axs[1,j].set_ylabel("predators")

        axs[0,j].set_title(title_list[j])


plt.show()
plt.close()
```



limit cycle
stable spiral
stable f.p.
preys
preys
preys
predators
predators
predators

[ ]: