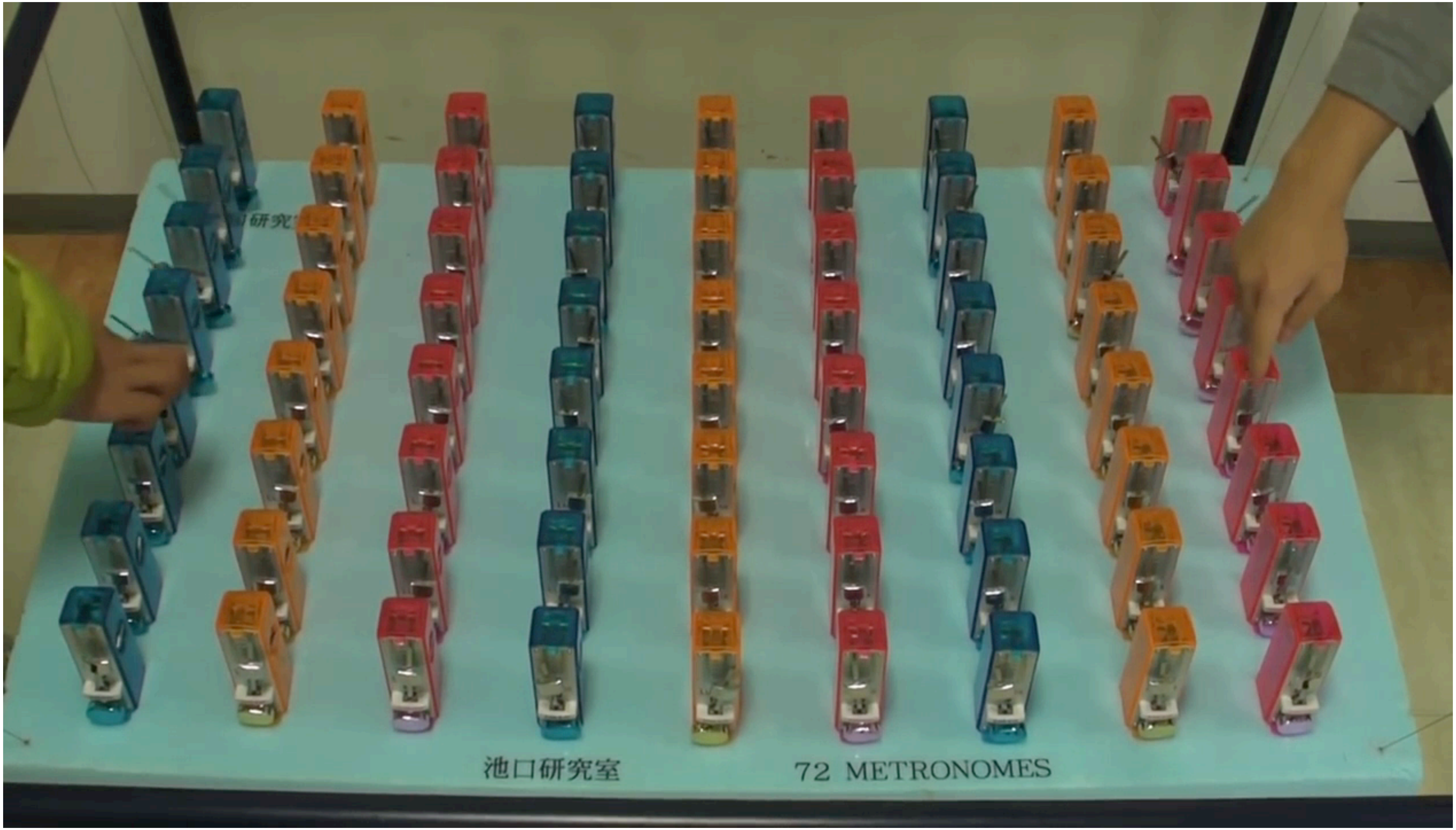
Synchronization in populations of phase oscillators

Kuramoto model
(Yoshiki Kuramoto, 1984)

Teaching example (BIO-341). The Kuramoto model

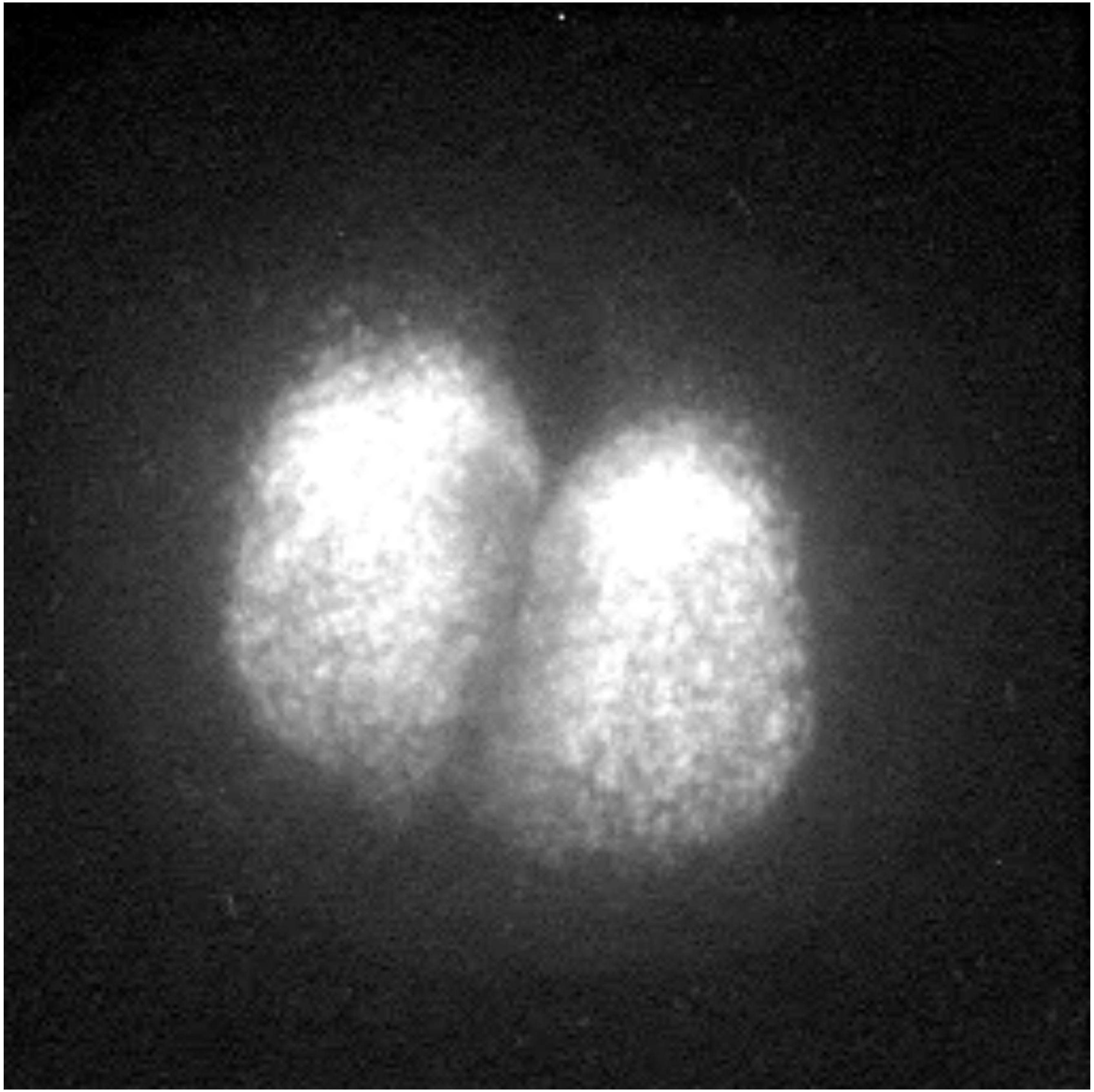
2 Modèle de Kuramoto



source: https://www.youtube.com/watch?v=suxu1bmPm2g

Fireflies resonate/synchronize too

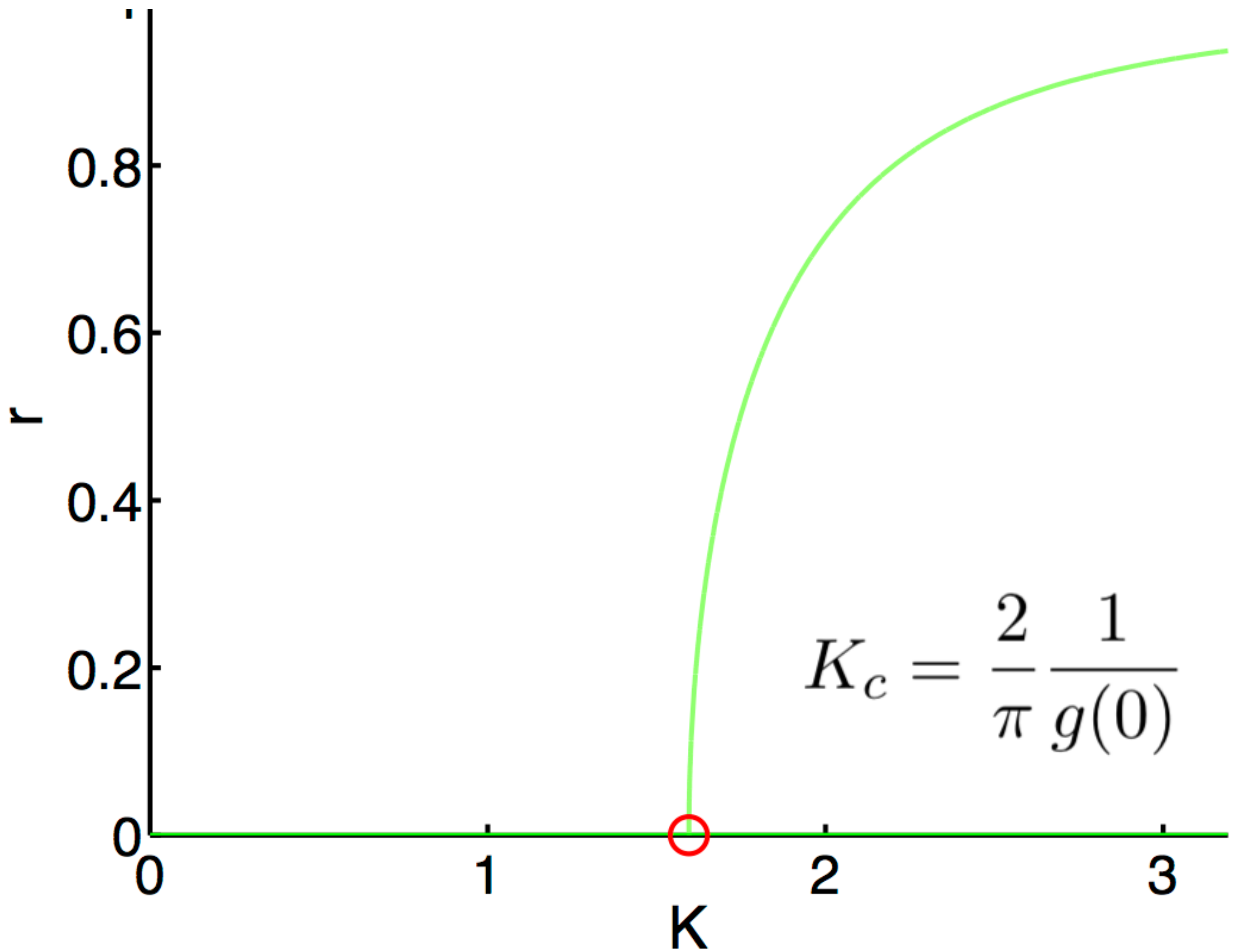
The mammalian master clock in the brain (SCN slice in mouse)



courtesy of Mick Hastings

The famous Kuramoto model

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i \in \{1, \ldots, N\}$$



The famous Kuramoto model
(summary of key steps)

All-to-all phase coupling model

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i \in \{1, \ldots, N\}$$

Distribution of frequencies

$\omega_i \sim g(\omega)$, $g(-\omega) = g(\omega)$
probability distribution
for example $g(\omega) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$

Rewrite in terms of the 'mean amplitude' r.

$r = \sum_i e^{i\theta_i} \longrightarrow$ "$\int e^{i\theta(\omega)} g(\omega)\, d\omega$"

$$\dot{\theta}_i = \omega_i - rK\sin(\theta_i)$$

What is the value of *r* in function of K and ? Let's just calculate it

$$r = \langle e^{i\theta} \rangle_s + \langle e^{i\theta} \rangle_d$$

e: entrained, d: drifting

interesting part

= 0 due to symmetry (cf Notes)

That's the result

$$r = \langle \cos(\theta) \rangle_s = rK \int_{-\pi/2}^{\pi/2} \cos^2(\theta) g(Kr\sin(\theta)) d\theta$$

- Implicit relation between r, K, and g
- r=0 is always a solution (branch 1)
- The equation has a 2nd branch r(K). What can we say about this second branch?
  - For r=0, K = Kc $\quad K_c = \frac{2}{\pi}\frac{1}{g(0)}$

Gaussian $K_c = \frac{2\sqrt{2}}{\sqrt{\pi}} \cdot \sigma$

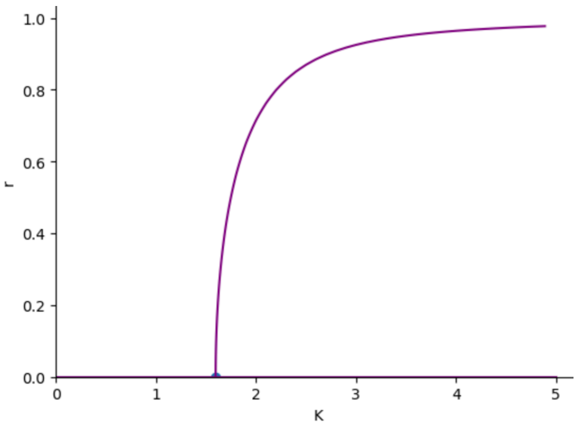
Let's check numerically for the Gaussian g()

```python
def Gaussian(x, mu=0, sig=1):
    return np.exp(-(x-mu)*(x-mu) / (2*sig*sig)) / (np.sqrt(2*np.pi)*sig)

def integrand(x,K,r,sig):
    return np.cos(x)**2 * Gaussian(K*r*np.sin(x),0,sig)

def integral(K,r,sig):
    return quad(integrand, -np.pi/2, np.pi/2, args=(K,r,sig))

def rel(K,r, sig):
    return K*integral(K,r,sig)[0]-1
```

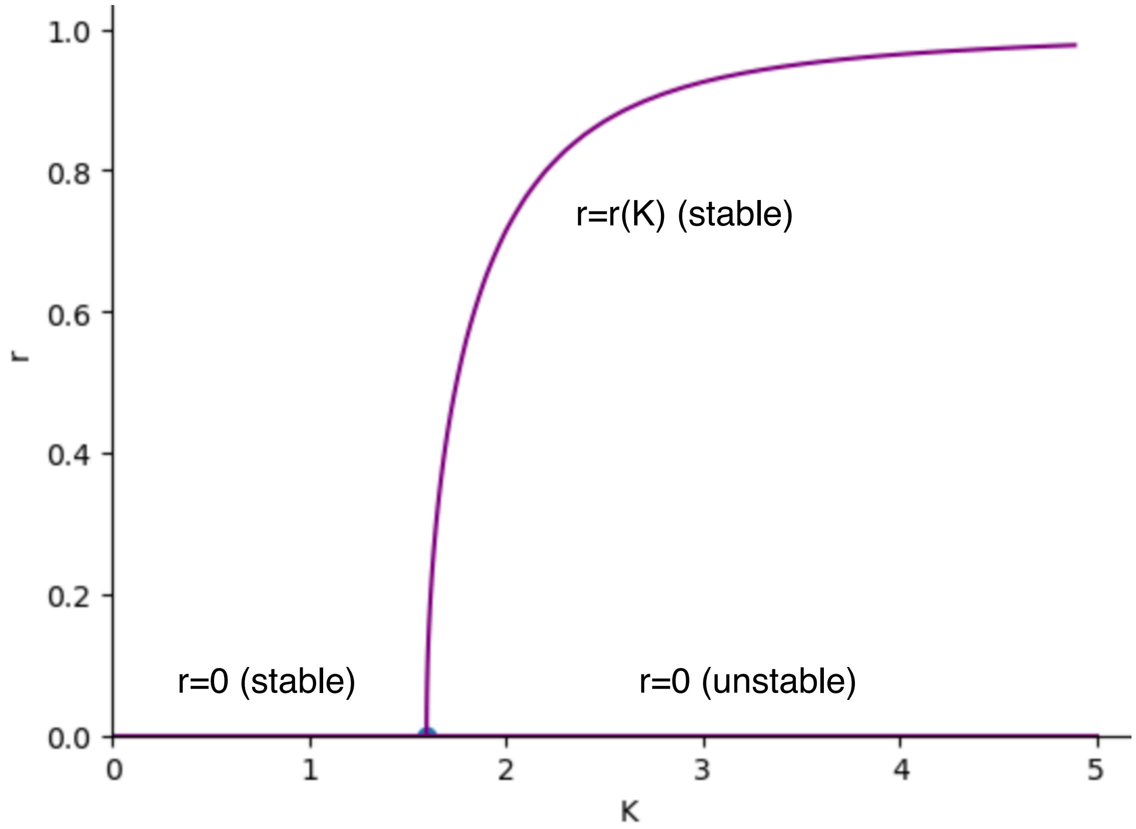



```python
sigma_w = 1
Kc = 2 / np.pi * np.sqrt(2*np.pi) * sigma_w

grid_r=np.arange(0,1.1,0.01)
grid_K=np.arange(0,5,0.1)
KK,rr=np.meshgrid(grid_K,grid_r)
V=np.array([0.])
Z=np.zeros_like(KK)
for i in range(KK.shape[0]):
    for j in range(KK.shape[1]):
        Z[i,j]=rel(KK[i,j],rr[i,j], sigma_w)
plt.contour(KK,rr,Z,V, colors='purple')
plt.plot([0,5],[0,0], color='purple')
plt.scatter(Kc,0)
plt.xlabel('K')
plt.ylabel('r')
plt.show()
```

Try with other distributions (see Notebook on Moodle)!

The two branches



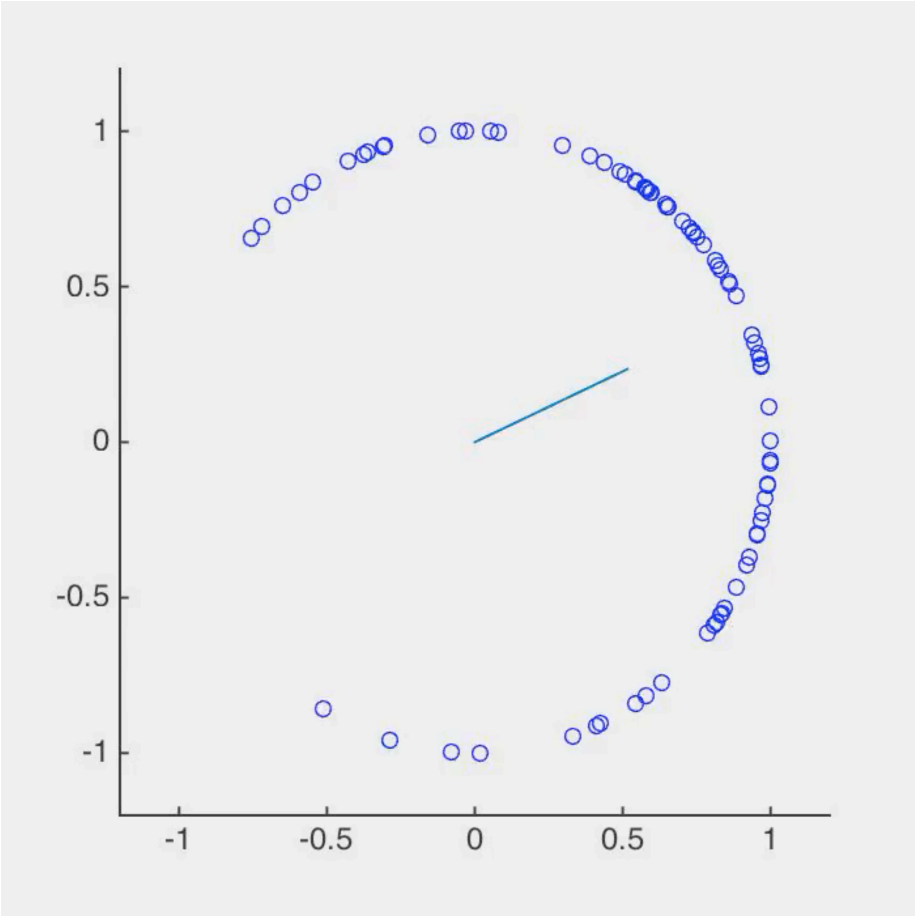
Think about what happens if you increase K from 0?

How about stability of the two solutions?

- Stability calculations are much harder since we have in infinite-dimensional SD (more advanced course)
- The easiest is to proceed numerically (simulations)

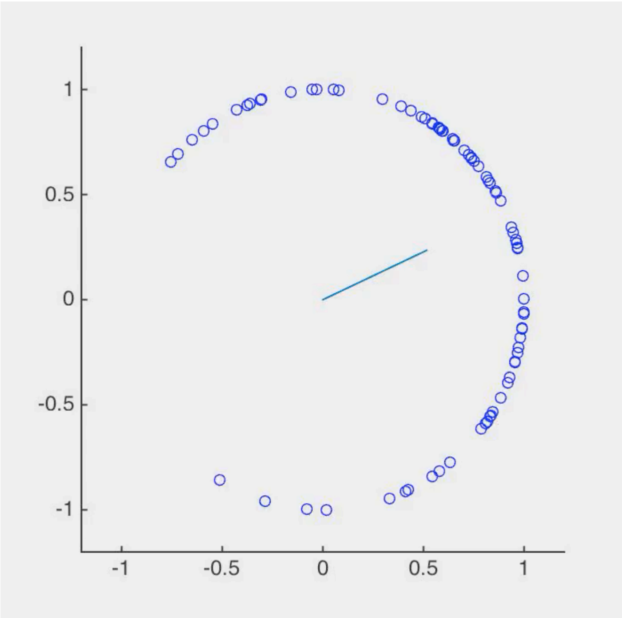
Simulating the Kuramoto model

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i \in \{1, \ldots, N\}$$

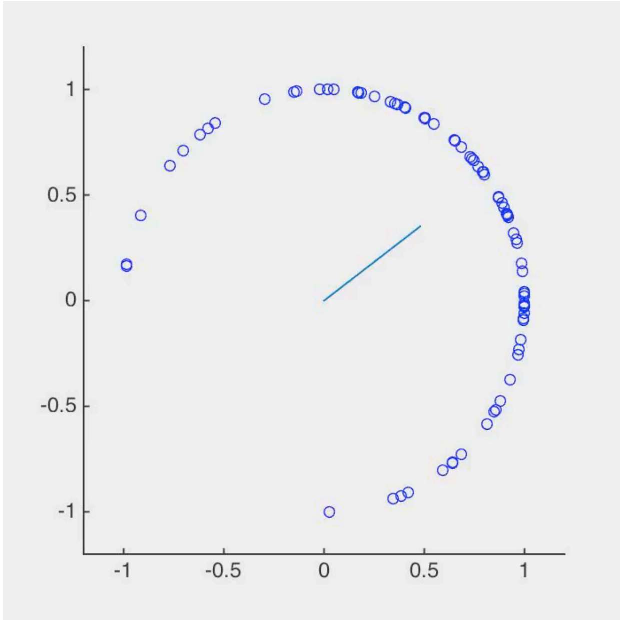


K=0.2*Kc

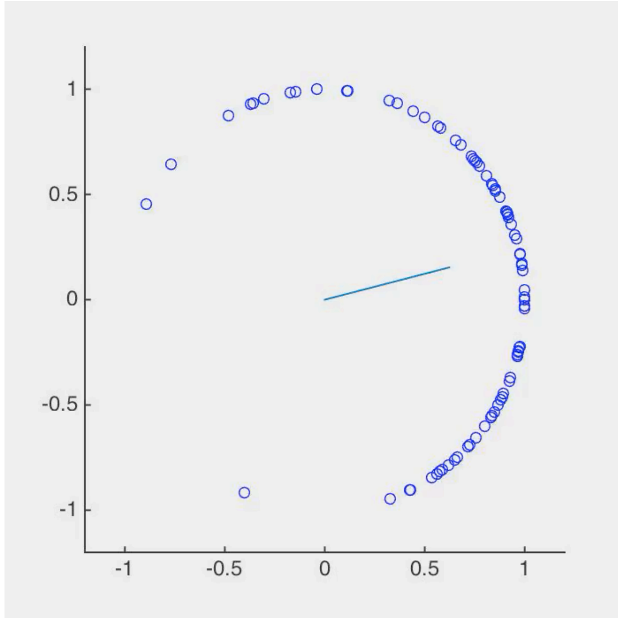
Kuramoto model



K=0.2*Kc



K=2*Kc



K=3*Kc

Kuramoto-like models

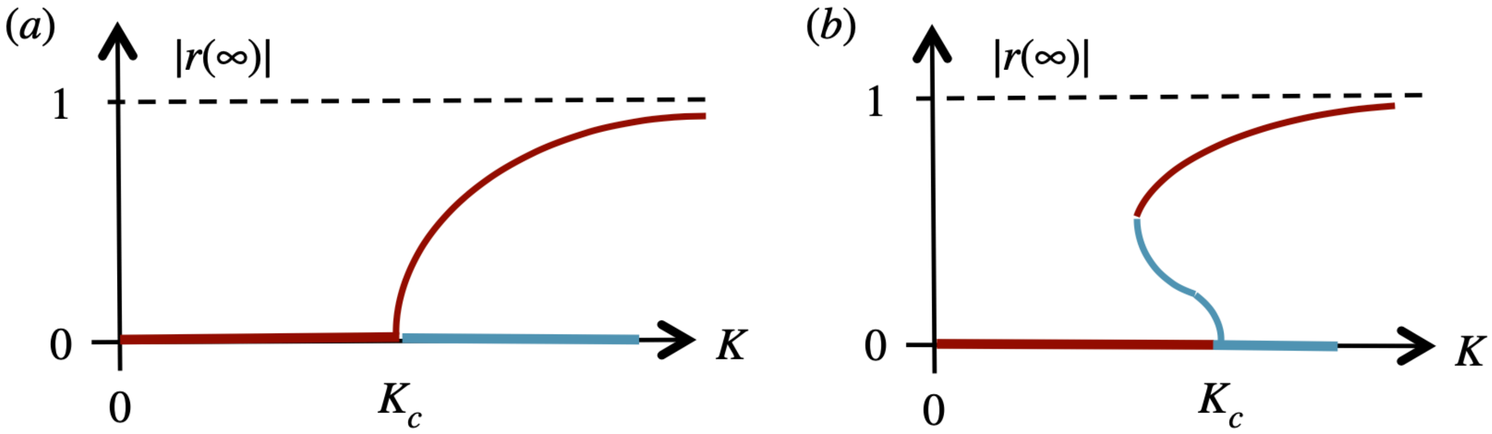


Figure 1. Schematic bifurcation diagram (*a*) for a symmetric and unimodal frequency distribution g and (*b*) for the bi-Cauchy distribution $g_{\Delta,\Omega}$ when bimodal (see §5). Red (resp. blue) lines indicate stable (resp. unstable) stationary solutions, see text for details.

Unimodal $g(\omega)$

$$g_{\Delta,\Omega}(\omega) = \frac{\Delta}{2\pi}\left(\frac{1}{(\omega-\Omega)^2+\Delta^2}+\frac{1}{(\omega+\Omega)^2+\Delta^2}\right)$$

Experiment with the notebook to obtain those diagrams