Lecture 9: Genetic Control System

Chapter 7.2.2, Strogatz ch. 8

11/11/24 $\dot{x} = -ax + y$ $x = $ [Protein]

$$\dot{y} = -by + \frac{x^2}{1+x^2}$$ $y = $ [RNA] coding for protein x

$a, b > 0$ are parameters relating to decay/lifetime of protein/RNA

Identify the terms:

$-ax, -ay$ linear decay NB First quadrant only

$+y$ production of protein from RNA

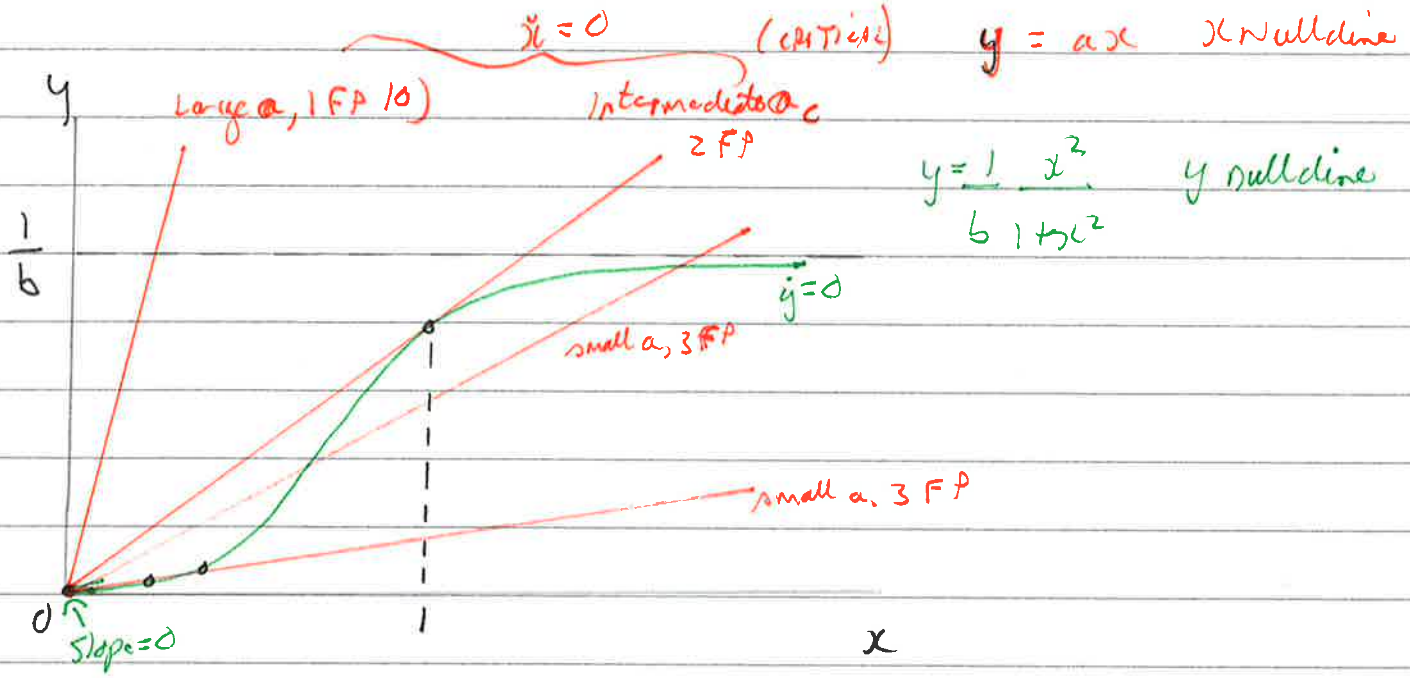
$\frac{x^2}{1+x^2}$ autocatalytic feed back from protein to RNA; it saturates because there is a limited number of binding sites for promoter.

cp. $\dot{g} = s - rg + \frac{g^2}{1+g^2}$ But now we separate the RNA and protein dynamics.

Nullclines

$\dot{x} = 0 \Rightarrow$ $\underline{\underline{y = ax}}$ i.e. linear

$\dot{y} = 0 \Rightarrow$ $\underline{\underline{by = \frac{x^2}{1+x^2}}}$



As $x \to \infty$ $y \to 1/b$ What is $\frac{dy}{dx}(x=0)$? $= \frac{2x}{(1+x^2)^2} = 0$

Clearly, (0,0) is always a fixed point.

For a less than a certain value, there are 3 FPs, and for a special value, there are two.

Where are the non-trivial FPs? Intersection of nullcline.

$$a x^* = \frac{1}{b}\frac{x^2}{1+x^2}$$

$\Rightarrow x^* = 0$ or $ab = \frac{x^*}{1+x^{*2}}$ i.e. a quadratic equation in x^* so possibly 0, 1 or 2 real roots.

(Drop the * now) Non-trivial fixed points

$ab/(1+x^2) = x$

$$\Rightarrow \quad x^2 - \frac{x}{ab} + 1 = 0$$

$$\therefore x = \frac{\frac{1}{ab} \pm \sqrt{\left(\frac{1}{ab}\right)^2 - 4}}{2} = \frac{1}{2ab}\left[1 \pm \left(1 - (2ab)^2\right)^{\frac{1}{2}}\right]$$

Non-zero fixed point

So, if $2ab < 1$, there are two non-trivial fixed points (as well as $0,0$)

$2ab = 1$, there is one non-trivial FP : $x^* = 1$, $y^* = a$

$2ab > 1$, only $(0,0)$ is a fixed point.

Jacobian

$$J = \begin{pmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{pmatrix}$$

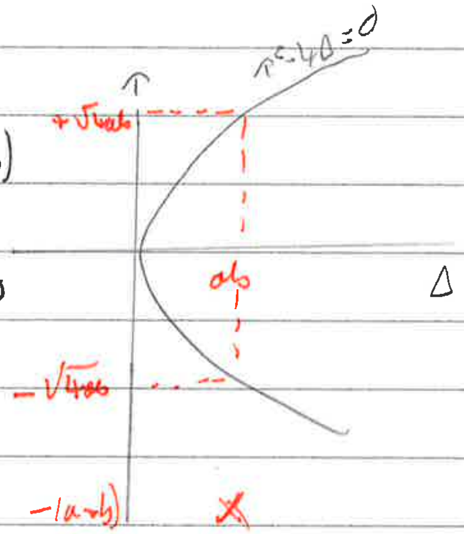
$\boxed{\therefore \tau = -(a+b) < 0}$ as $a, b > 0$ $\Rightarrow$ All fixed points are stable or saddlepoints

$$\boxed{\Delta = ab - \frac{2x}{(1+x^2)^2}}$$

Analyse each Fixed point

A) $(0,0)$ $\quad \underline{\underline{J}} = \begin{pmatrix} -a & 1 \\ 0 & -b \end{pmatrix}$ $\quad \therefore \tau = -(a+b)$

$\Delta = ab > 0$



But $\tau^2 - 4\Delta = (a+b)^2 - 4ab$

$= (a-b)^2 > 0$

$\therefore (0,0)$ is a stable node for all values of a, b. NB Ignoring $a = 0$!

Before we look at the non-zero fixed points, we need some results.

For $a > a_c$, the origin is the only fixed point. When $a = a_c$, we have two fixed points, and for $a < a_c$, we have three.

The bifurcation occurs when the nullclines are tangent to each other.

1) $\Rightarrow$ Function values are equal

2) and Slope of functions (tangents) are equal

1) $\Rightarrow$ $\boxed{ab = \dfrac{x}{1+x^2} \quad \text{from before}}$

2) $\Rightarrow$ $a = \dfrac{1}{b}\left[\dfrac{(1+x^2)\cdot 2x - x^2 \cdot 2x}{(1+x^2)^2}\right] = \dfrac{2x}{b(1+x^2)^2}$

$$\boxed{\therefore ab = \frac{2x}{(1+x^2)^2}}$$

Divide one by the other:

$$1 = \frac{1+x^2}{2} \Rightarrow x = 1$$ (Draw this on phase portrait)

and so, also from $ab = \frac{x}{1+x^2} = 1/2$

$$\boxed{\therefore\ 2a_c b = 1} \qquad a_c = \text{critical value of } a$$

Where are the non-zero fixed points for $2ab < 1$?

B) $x_1^* = \frac{1}{2ab}\left[1 - (1-(2ab)^2)^{\frac{1}{2}}\right]$

we know that $\tau = -(a+b)$, what is Δ?

$$\Delta = ab - \frac{2x_1^*}{(1+x_1^{*2})^2}$$ This looks hard but we know $ab = \frac{x_1}{1+x_1^{*2}}$

$$\therefore \Delta = ab - \frac{2}{(1+x_1^{*2})} \cdot \underbrace{\frac{x_1^*}{(1+x_1^{*2})}}_{\equiv ab}$$

$$= ab\left[1 - \frac{2}{(1+x_1^{*2})}\right] = ab\left[\frac{1+x_1^{*2} - 2}{1+x_1^{*2}}\right] = ab\left[\frac{x_1^{*2} - 1}{x_1^{*2}+1}\right] < 0$$

Q. Why is $x_1^* < 1$? $\qquad$ ∴ This is a saddle point as $\Delta < 0$.

How do we know $x_1^* < 1$ for $x \in [0, 1]$

Consider $f(x) = \dfrac{1 - (1-x^2)^{\frac{1}{2}}}{x}$ i.e. Let $x = 2ab$ in Eqn B

Consider the product: $\left(\dfrac{1 - (1-x^2)^{\frac{1}{2}}}{x}\right) \times \left(\dfrac{1 + (1-x^2)^{\frac{1}{2}}}{x}\right)$

$$= \frac{1 - (1-x^2)}{x^2}$$

$$= 1$$

If two numbers multiplied together give 1, one must be larger than 1, and the other smaller than 1.

Also $1 - (1-x^2)^{\frac{1}{2}} < 1 + (1-x^2)^{\frac{1}{2}}$

$$\therefore \frac{1 - (1-x^2)^{\frac{1}{2}}}{x} < 1 < \frac{1 + (1-x^2)^{\frac{1}{2}}}{x}$$

i.e. The two non-zero roots are on opposite sides of 1.

c) $x_2^* = \frac{1}{2ab}\left[1 + (1 - (2ab)^2)^{\frac{1}{2}}\right]$

again, we know $\tau < 0$, what is Δ?

$\Delta = ab - \frac{2x_2^*}{(1 + x_2^{*2})^2}$ but again $\frac{x_2^*}{1 + x_2^{*2}} = ab$

$\Rightarrow \Delta = ab\left[\frac{x_2^{*2} - 1}{x_2^{*2} + 1}\right]$ so numerator < denominator

$\therefore \Delta < ab$

$\Rightarrow \tau^2 - 4\Delta > (a+b)^2 - 4ab = (a-b)^2 > 0$

$\therefore x_2^*$ is a stable node

so, we have a stable node at $(0,0)$ and $(x_2^*, a x_2^*)$

and a saddlepoint at $(x_1^*, a x_1^*)$

if $2ab < 1$

What are the vector fields along the nullclines?

$\dot{x} = 0 \Rightarrow y = ax$

and $\dot{y} = -by + \frac{x^2}{1+x^2} = \frac{-abx + x^2}{1+x^2} = \frac{-abx(1+x^2) + x^2}{1+x^2}$

When does this change sign? only the numerator matters.

$$x^2 - abx(1+x^2) = 0$$

$\Rightarrow x = 0$ or $x - ab - abx^2 = 0$

$$\Rightarrow x^2 - \frac{x}{ab} + 1 = 0$$

which has solutions x_1^*, x_2^* as before.

$\Rightarrow \dot{y} \sim -abx(x - x_1^*)(x - x_2^*)$

NB cubic term must be negative because it is negative above in original equation for $\dot{y}$

$\therefore \dot{y} < 0$ when $x < x_1^*$ and $x > x_2^*$

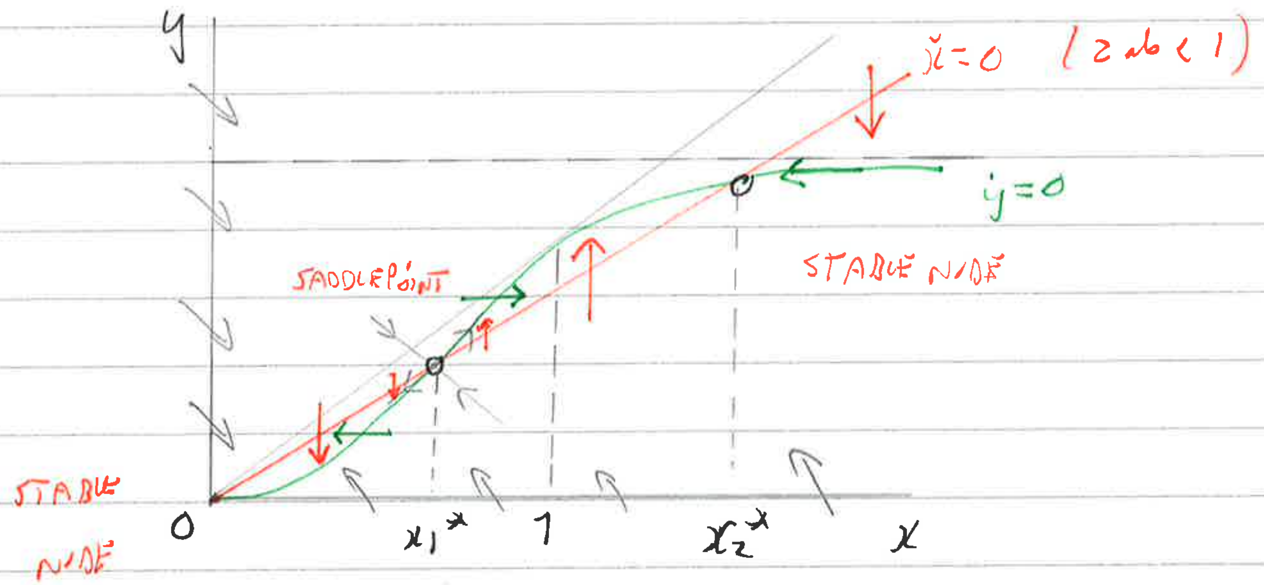
$\dot{y} > 0$ when $x_1^* < x < x_2^*$

$$\dot{y} = 0 \Rightarrow by = \frac{x^2}{1+x^2}$$

and $\dot{x} = -ax + y = -ax + \frac{1}{b} \cdot \frac{x^2}{1+x^2} = \frac{1}{b}\left[\frac{-ab\,x(1+x^2) + x^2}{1+x^2}\right]$

$\therefore$ $\dot{x} < 0$ when $x < x_1^*$ and $x > x_2^*$ — same as $\dot{y}$ except for constant b

$\dot{x} > 0$ when $x_1^* < x < x_2^*$.



what is $\dot{x}$ when $x = 0$? $\dot{x} = y > 0$ } all up the y axis
and $\dot{y}$? $\dot{y} = -by < 0$

Recall $\dot{x} = -ax + y$

$$\dot{y} = -by + \frac{x^2}{1+x^2}$$

and along $y = 0$? $\dot{x} = -ax < 0$

$\dot{y} = \frac{x^2}{1+x^2} > 0$ } all along the x axis

Numerical example

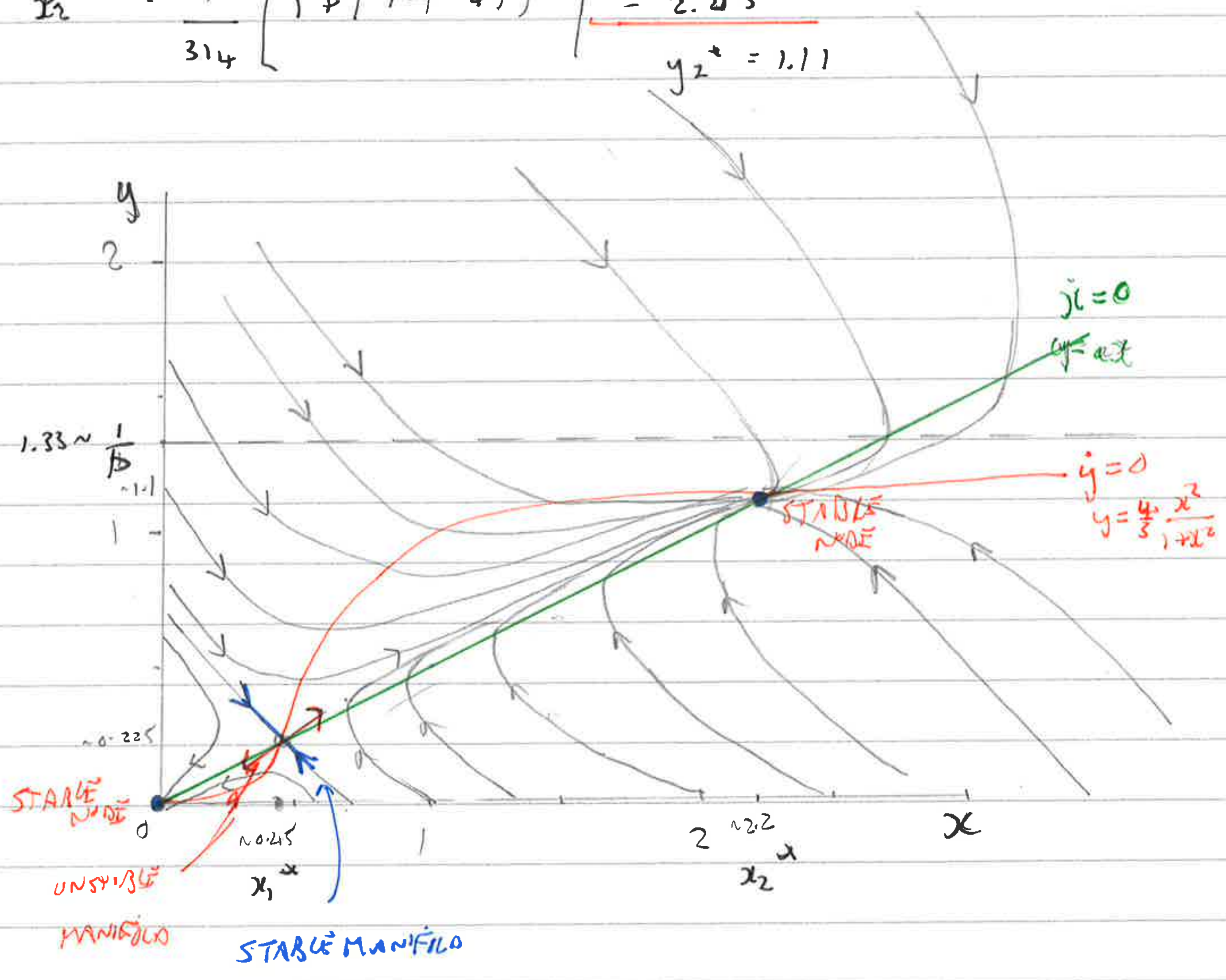
we need $2ab < 1$, try $a = 1/2$, $b = 3/4$ $\Rightarrow 2ab = 3/4$

$$x_1^* = \frac{1}{3/4}\left[1 - \left(1 - (3/4)^2\right)^{\frac{1}{2}}\right] = \frac{4}{3}\left[1 - \left(1 - \frac{9}{16}\right)^{\frac{1}{2}}\right] = 0.45$$

$\Rightarrow y_1^* = ax^* = 0.225$

$$x_2^* = \frac{1}{3/4}\left(1 + \left(1 - (3/4)^2\right)^{\frac{1}{2}}\right) = 2.25$$

$y_2^* = 1.11$



NOTE

1. Stable and unstable manifolds & fast and slow eigenvectors are straight lines only close to the fixed points.

2 Stable manifold divides the phase plane into distinct parts

3 For $a > a_c$ only $(0,0)$ is a fixed point

4 When $a < a_c$ a stable node appears to large [Protein], separated from $(0,0)$ by saddle points.

Lecture 9 GCS

Strogatz 8.1.1. (Example)

$2ab < 1$