Lecture 8

We have now seen all possible long-time behaviours of a 2D dynamical system: trajectories must:

- approach a fixed point (directly or spiral around it. NB. You never actually get there, only at time infinity or minus infinity.)
- or go to infinity
- or approach a limit cycle

- Limit cycles represent systems that can *oscillate* without an external driving force, e.g., heart contraction, chemical reactions, neurons
- The amplitude, frequency, and shape of a limit cycle are set by the equations not by the initial conditions; if perturbed, a system returns to the limit cycle (not so equilibrium reactions)
- How can fixed points/limit cycles be created/destroyed? Bifurcations (back to 1D for a while …)

Recap of lecture 7: Poincaré Bendixson theorem

A **Limit Cycle** is an **Isolated**, **Closed**, trajectory.

Isolated means that nearby trajectories are not closed (unlike centres), they either spiral into the limit cycle (a stable one) or they spiral away from it (an unstable one).

Closed means $O(t + T) = O(t)$ with a period T, so the trajectory goes round and round. This represents self-sustaining oscillations.

Poincaré-Bendixson theorem states that there is a limit cycle in a bounded region R if:

1) all trajectories point inwards (stable limit cycle) or outwards (unstable limit cycles) on the boundary of R (a **trapping** region)

2) there is an unstable (stable) node or spiral inside the trapping region to drive trajectories to (from) the limit cycle from within.

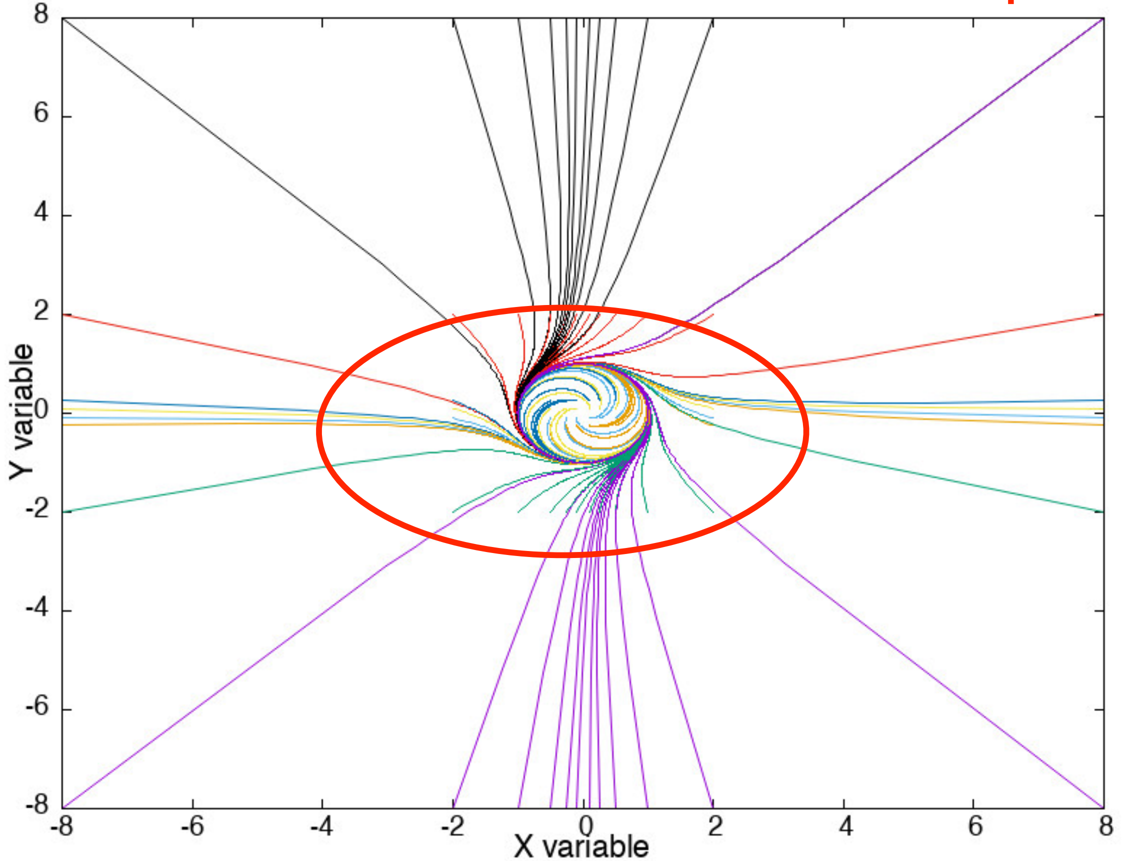
There cannot be a saddlepoint.

Simple limit cycle

$$dx/dt = -y + x\,(1 - x^2 - y^2)$$
$$dy/dt = x + y\,(1 - x^2 - y^2)$$

$$dr/dt = r\,(1 - r^2)$$
$$d\varphi/dt = 1$$

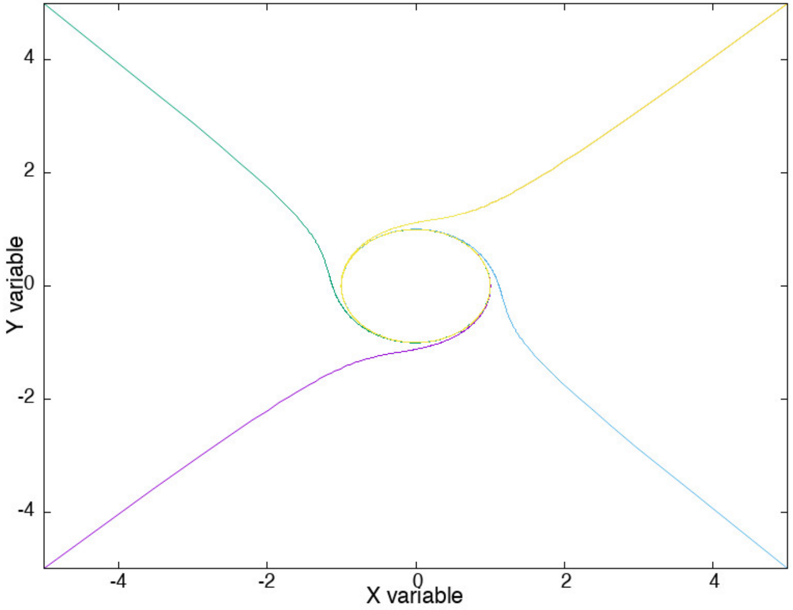
in plane polar coordinates



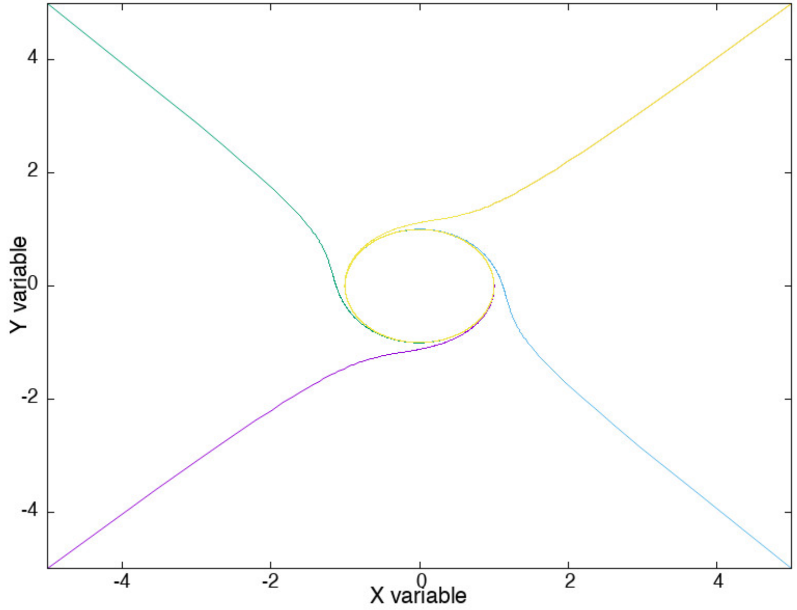
Trapping region for this case is any closed curve around the limit cycle

Why are trajectories at large x, y straight lines? Shouldn't they be rotating?

Close up



RK with 1000 points, dt = 0.01



10,000 points, dt = 0.001

$dr/dt = r (1 - r^2)$

$d\varphi/dt = 1$

For large r, $dr \sim -r^3 dt$ and is much greater than $d\varphi \sim dt$. Only when their magnitudes are comparable, do we see the curvature.

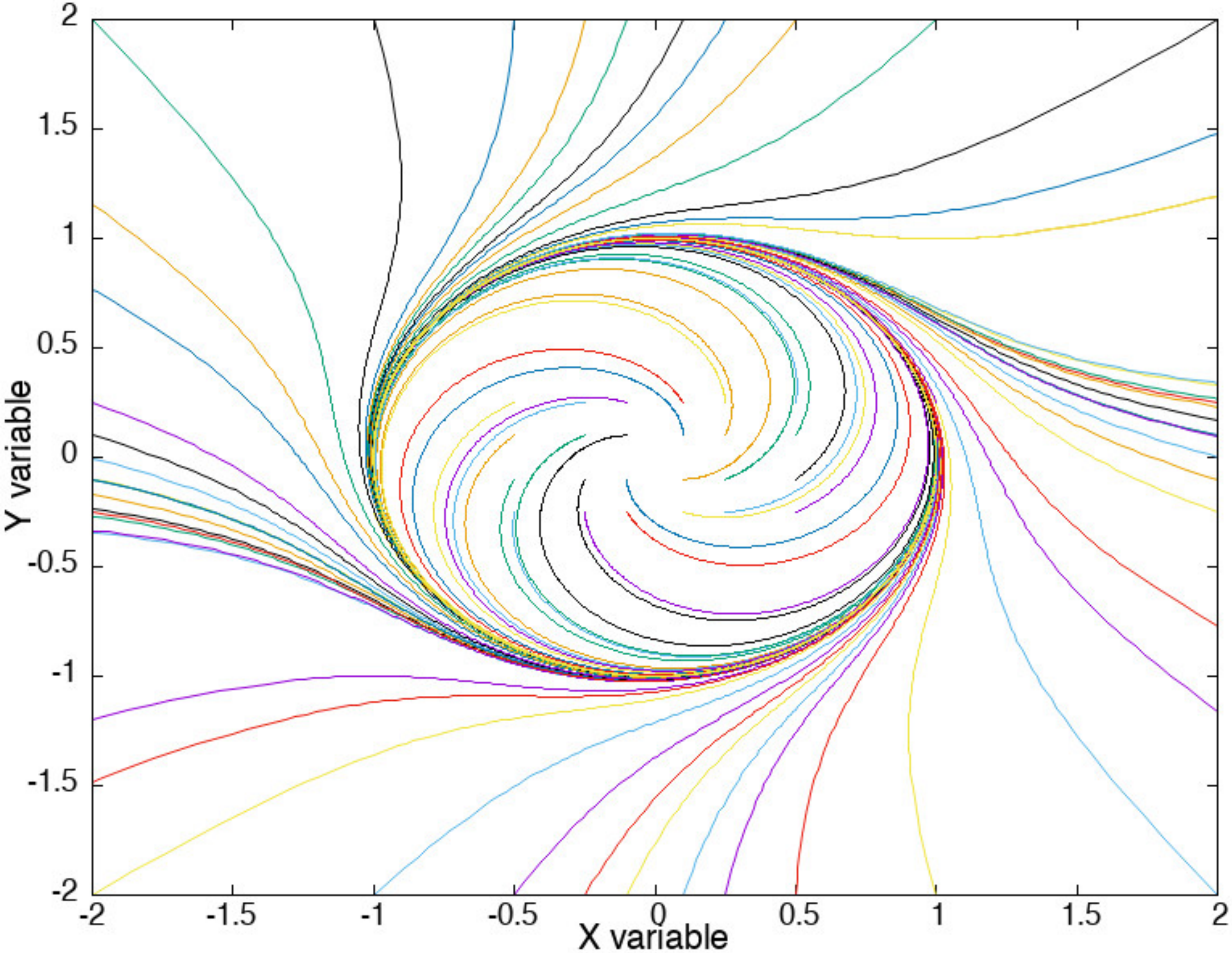
Note. Phase portraits only show the direction of trajectories not the speed along them.

Where are the nullclines? They're clearly not the axes.

$dx/dt = -y + x (1 - x^2 - y^2) = 0$
$dy/dt = x + y (1 - x^2 - y^2) = 0$

They're cubic equations, e.g.

$y^3 - (1 - x^2) y - x = 0$ for $dy/dt = 0$

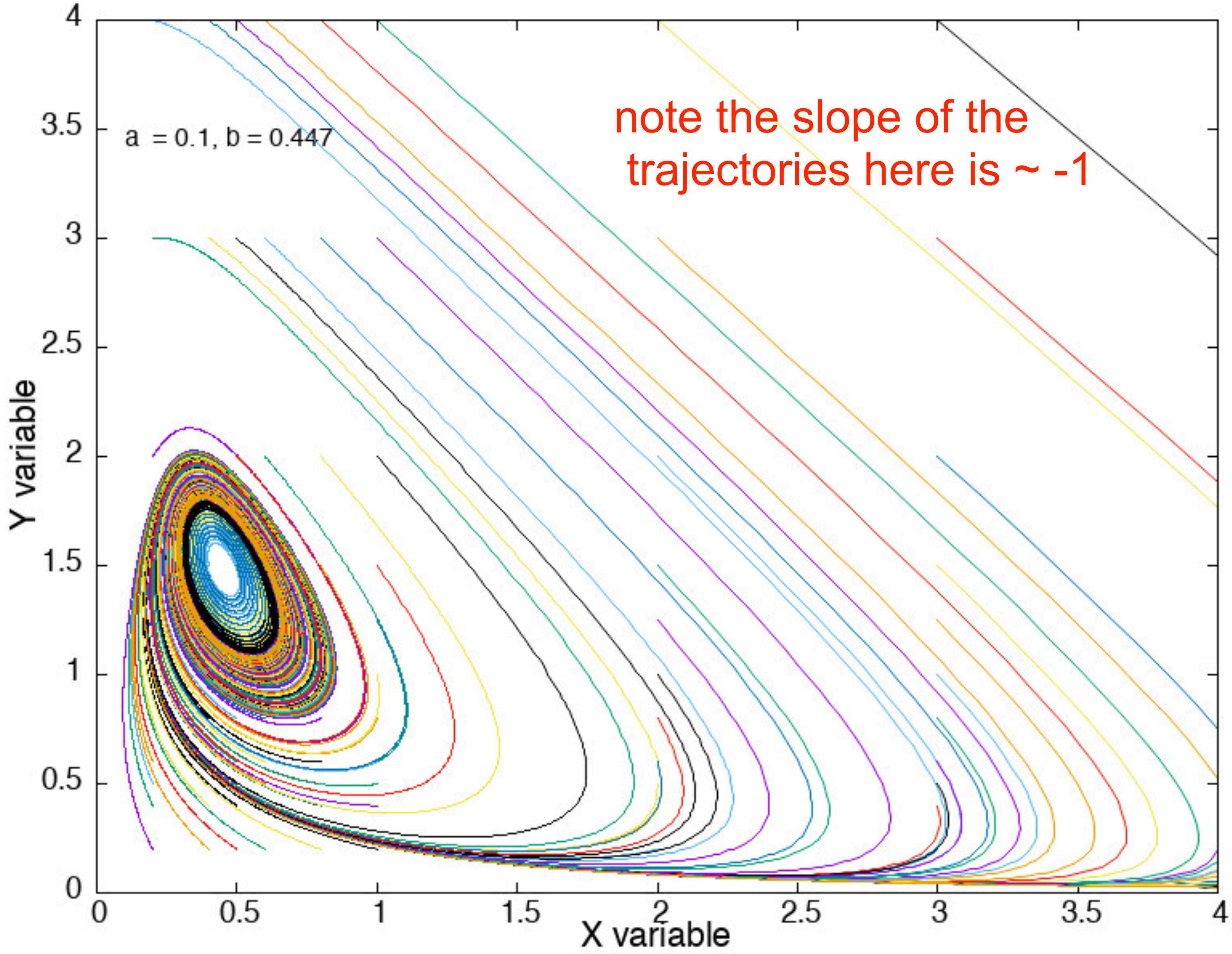


Go through:

(+/-1, 0) for x

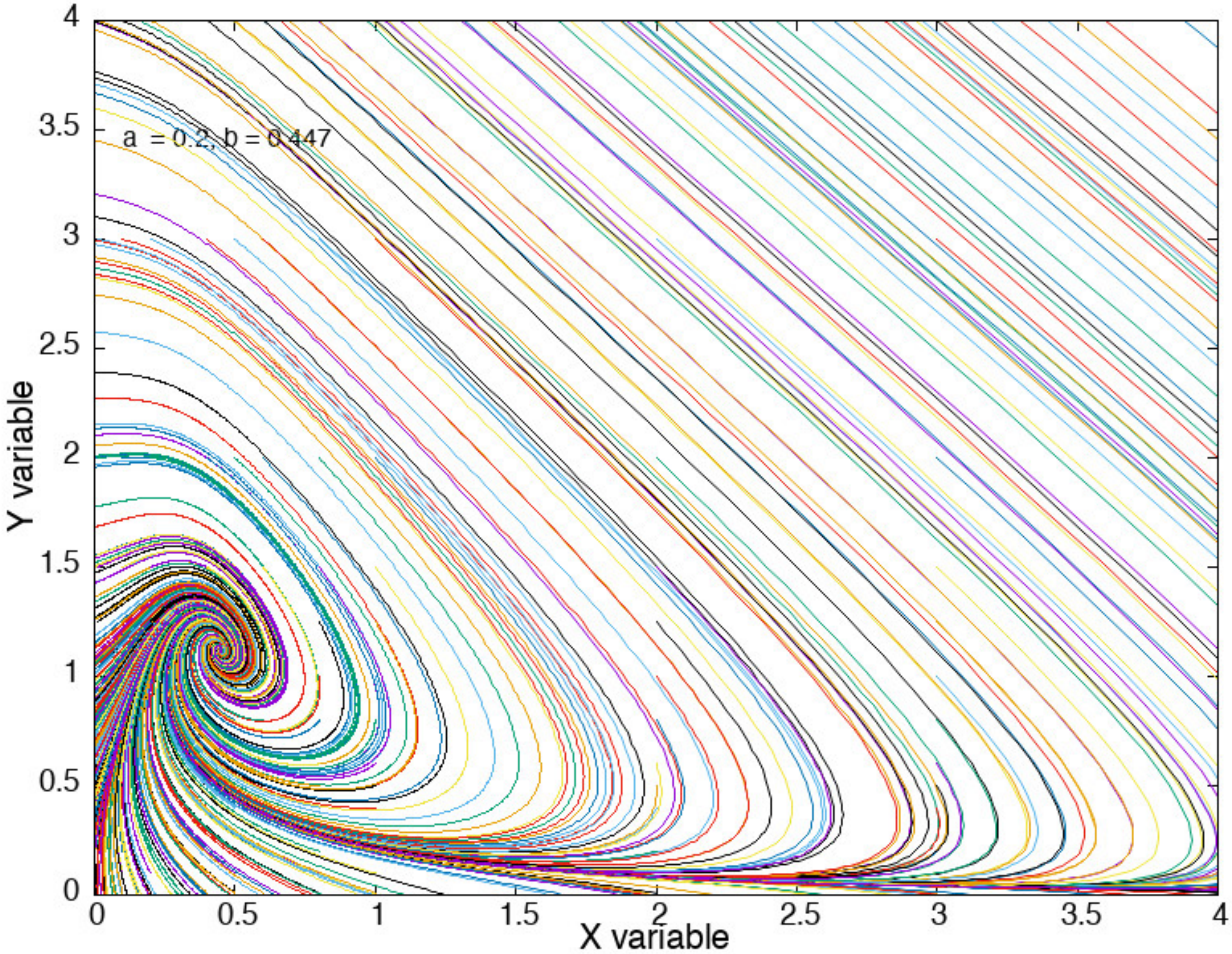
(0, +/- 1) for y

Selkow model with: a = 0.1, b = 0.447
Fixed point at (0.447, 1.49) is a stable limit cycle



a = 0.2, b = 0.447

Fixed point at (0.447, 1.12) is now a stable spiral



Background quiz: go.epfl.ch/turningpoint

Session Id: julian23

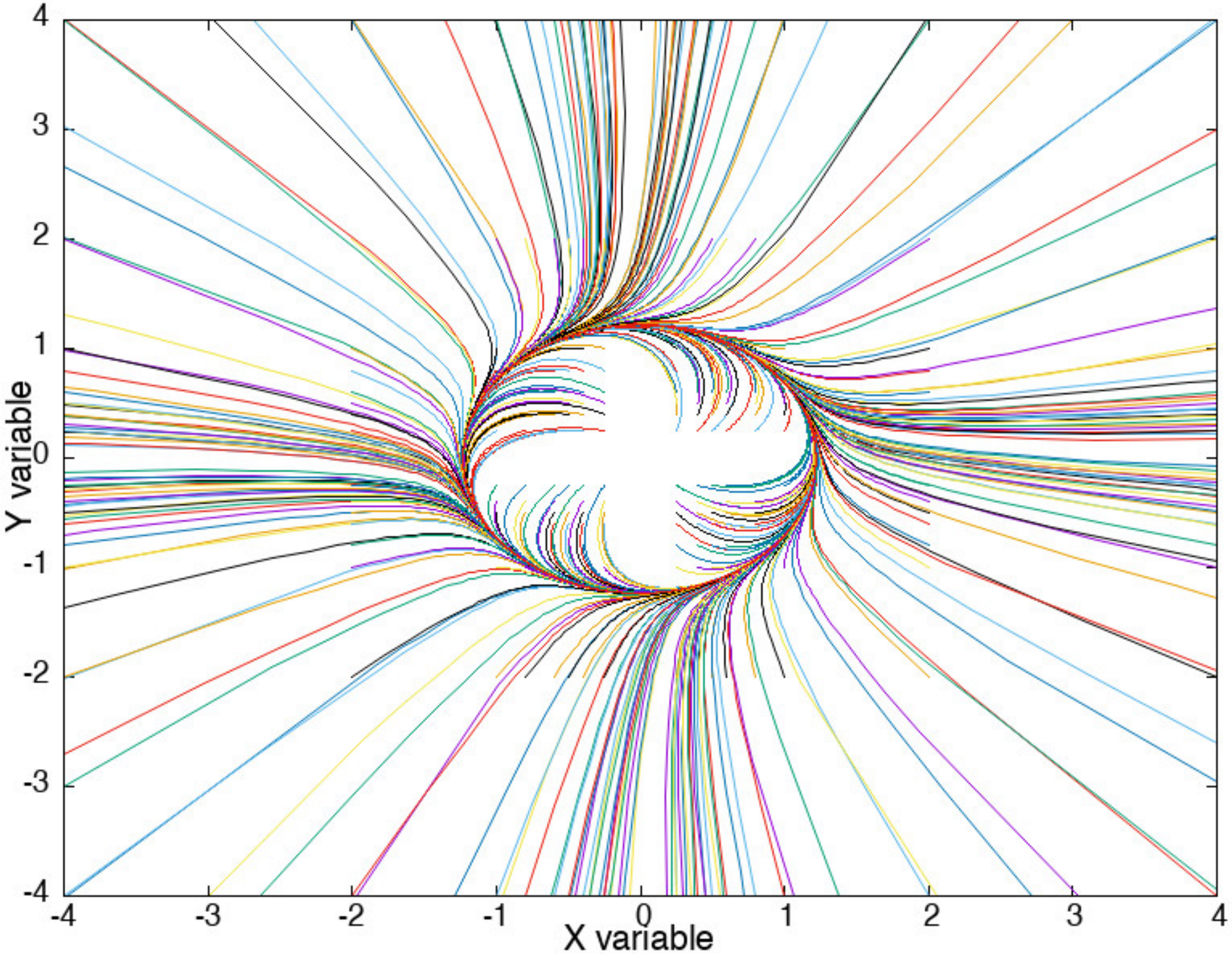
All input is anonymous; data are stored outside CH

Break

I was playing with the equations, trying to find 2 limit cycles

and …

$$dx/dt = -y + x(1 - x^2 - y^2) + (x - 4)(1 - x^2 - y^2)$$
$$dy/dt = x + y(1 - x^2 - y^2) + (y - 4)(1 - x^2 - y^2)$$

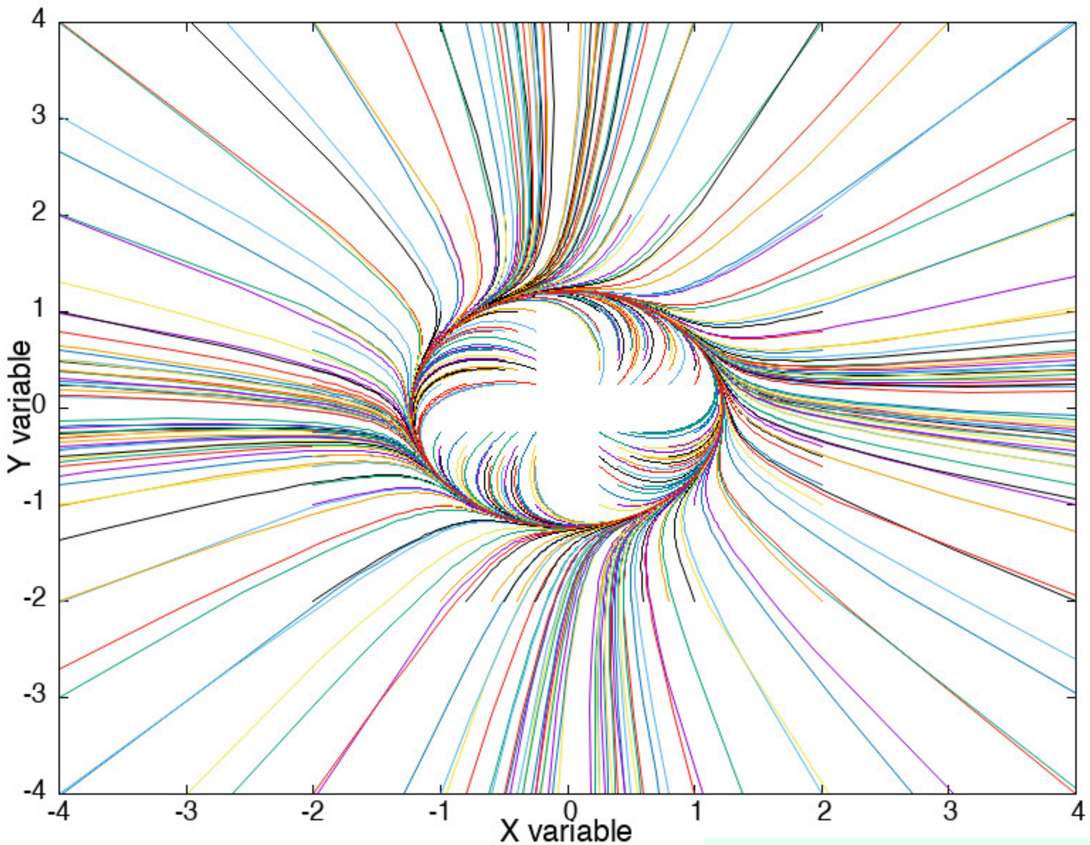


Challenge: Can you find 2 limit cycles?

… starting from …

Edible prize

$$\frac{dx}{dt} = -y + x(1 - x^2 - y^2) + \ldots$$
$$\frac{dy}{dt} = x + y(1 - x^2 - y^2) + \ldots$$



First 3 people/groups who send me the phase portrait and the equations

Tricky Points

In the tau-delta plot, if we are in the large, open areas, changing a parameter won't change a dynamical system's behaviour.

But if we are near a border … it can have catastrophic effects. Oscillatory behaviour can detach from a stable state, and blow up. You have to examine the full non-linear equations, NOT use the tau-delta plot.

Bifurcations

A bifurcation is a qualitative change in a system's behaviour as a parameter is changed by a small amount.

Fixed points can appear / disappear / change their type/stability.

There are only certain ways that FPs change; they occur in all dimensions, but 1D is easiest to visualise.