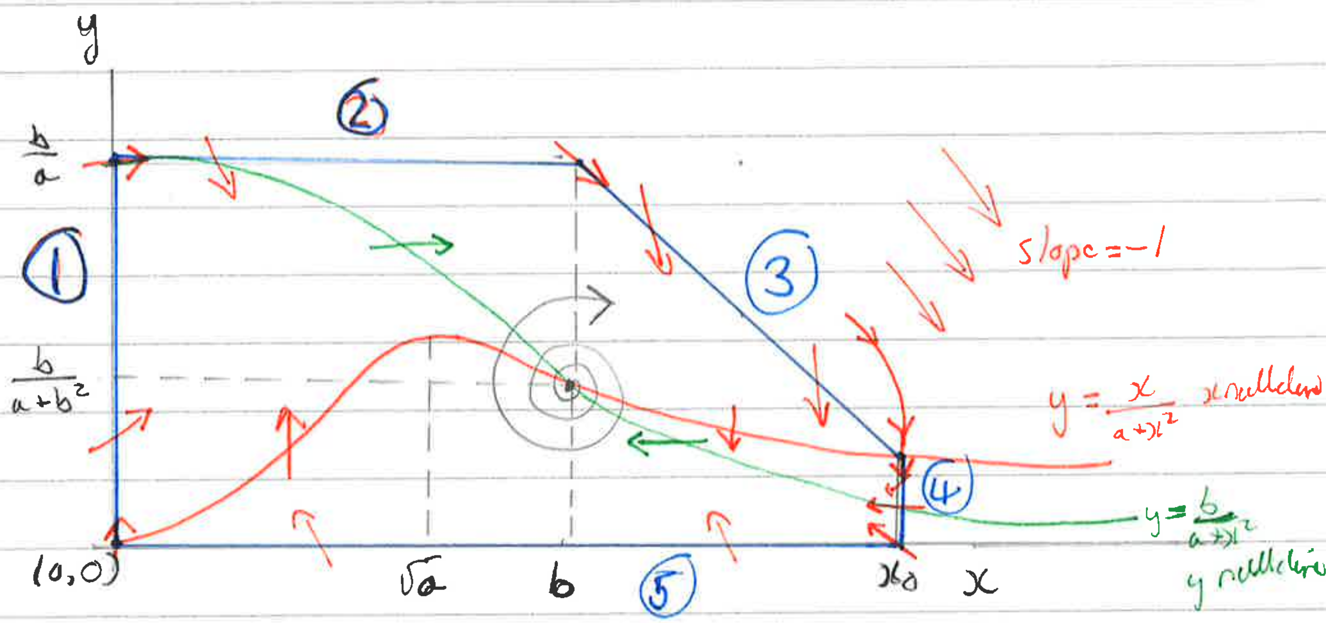
How do we show that all trajectories point inwards?

4/11/24



$$\dot{x} = -x + (a+x^2)\cdot y \qquad x, y \geq 0$$

$$a, b > 0$$

$$\dot{y} = b - (a+x^2)\cdot y$$

NB Note assumption $\sqrt{a} < b$

Why choose these regions? Axes are a good guess, especially for concentrations / populations.

Region 2 follows because the y nullcline starts at $y = b/a$; $x = 0$.
Regions 3, 4 are result of trial and error!

Region 1 (Don't forget corners)

origin: $\dot{x} = 0$
$\dot{y} = b > 0 \quad \therefore$ y axis

as soon as $y > 0$: $\dot{x} = ay > 0$
$\dot{y} = b - ay > 0$ until $y = b/a$

$y = b/a$: $\dot{x} = b$
$\dot{y} = 0$ $\quad \therefore$ Horizontal (NB. If $y > b/a$, $\dot{y}$ is negative so still into region.)

$\boxed{\text{Region 2:}}$ Horizontal line from $(0, b/a)$ to $(b, b/a)$

we've done the point $(0, b/a)$, so consider $0 < x < b$.

$$\dot{x} = -x + (a+x^2)y = -x + (a+x^2).\frac{b}{a} = b - x + bx^2 > 0$$

$$\dot{y} = b - (a+x^2).\frac{b}{a} = -\frac{b}{a}x^2 < 0$$

into region

Point $(b, b/a)$ and region 3 later

$\boxed{\text{Region 4}}$: we choose $x_0 > b$, so that the slope of the line in region 3 is -1, and take the vertical line up to the x nullcline from $(x_0, 0)$.

$(x_0, 0)$: $\dot{x} = -x_0 < 0$

$\dot{y} = b - (a+x_0^2).0 = b > 0$

into region

up the vertical line:

$\dot{x} = -x_0 + (a+x_0^2).y$ but we know that $y < \frac{x_0}{a+x_0^2}$ (x nullcline)

≤ 0 $\Rightarrow y.(a+x_0^2) < x_0$

$\dot{y} = b - (a+x_0^2).y$ decreases as y increases from ϕ, goes through ϕ for $y = \frac{b}{a+x_0^2}$ (y nullcline!), and is negative for $\frac{b}{a+x_0^2} < y < \frac{x_0}{a+x_0^2}$

$\left(x_0, \frac{x_0}{a+x_0^2}\right)$:

$\dot{x} = 0$ (x nullcline)

$\dot{y} = b - x_0 < 0$

$\boxed{\text{Region 5}}$

we've done the origin and $(x_0, 0)$. Consider $0 < x < x_0$ and $y = 0$.

$$\left.\begin{aligned} \dot{x} &= -x < 0 \\ \dot{y} &= b > 0 \end{aligned}\right\} \text{into region}$$

$\boxed{\text{Region 3}}$

This is a straight line with slope -1 from $(b, b/a)$ to $(x_0, \frac{x_0}{a+x_0^2})$

Why?

consider $\dot{x}, \dot{y}$ for large (x, y) : $\left.\begin{aligned} \dot{x} &\to x^2 y \\ \dot{y} &\to -x^2 y \end{aligned}\right\} \Rightarrow \frac{dy}{dx} \to -1$

more precisely, along Region 3:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b - (a+x^2).y}{-x + (a+x^2).y} = -\left[\frac{(a+x^2).y - b}{(a+x^2).y - x}\right]$$

But we know $x > b$ by assumption, ~~and~~ $y > \frac{x}{a+x^2}$ (x nullcline),

so the denominator is always positive and ratio has magnitude greater than 1

$\therefore \frac{dy}{dx} < -1$ i.e. steeper than the line $\Rightarrow$ into region.

$$(b, b/a): \left.\begin{aligned} \dot{x} &= -b + (a+b^2).b/a = b^3/a \\ \dot{y} &= b - (a+b^2).b/a = -b^3/a \end{aligned}\right\} \Rightarrow \frac{dy}{dx} = -1$$, so along the boundary.

LECTURE 8: BIFURCATIONS

(course book: CH7 ; Strogatz CH3)

2/8/24 BIFURCATION = A qualitative (topological) change in the dynamics of a system as a parameter is changed.

i.e. A fixed point appears/disappears / changes stability

A closed orbit " " "

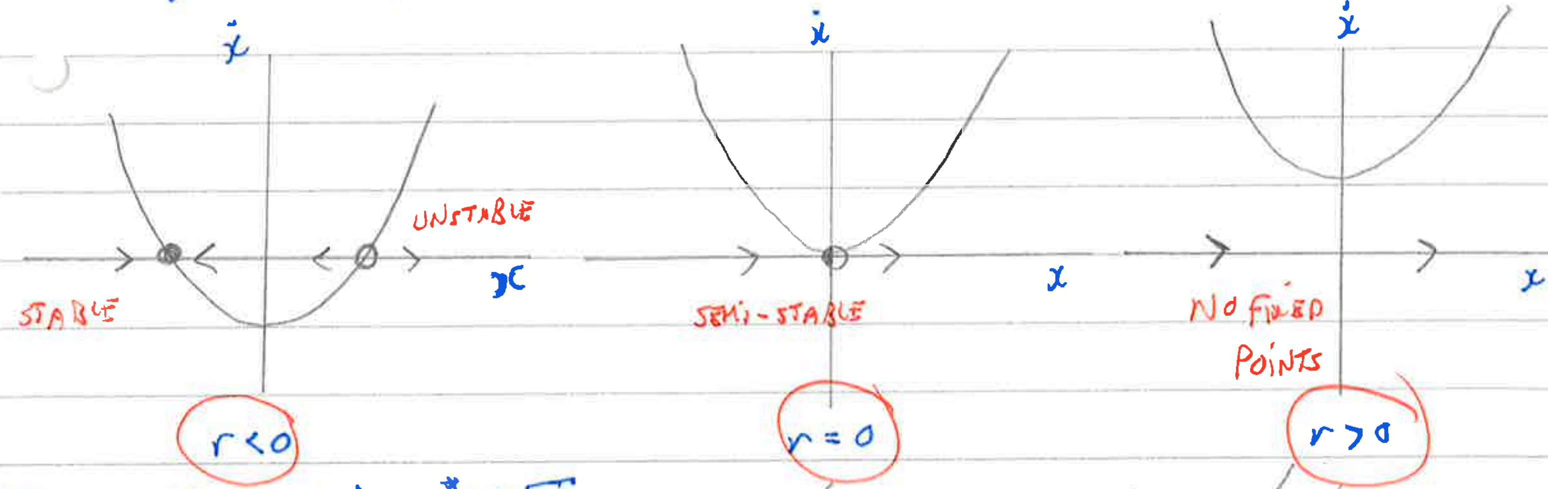
We consider first 1D bifurcations because even in higher dimensions, the same behaviour is observed restricted to a 1D subspace of the 2D, or higher, space.

Bifurcations in 1D

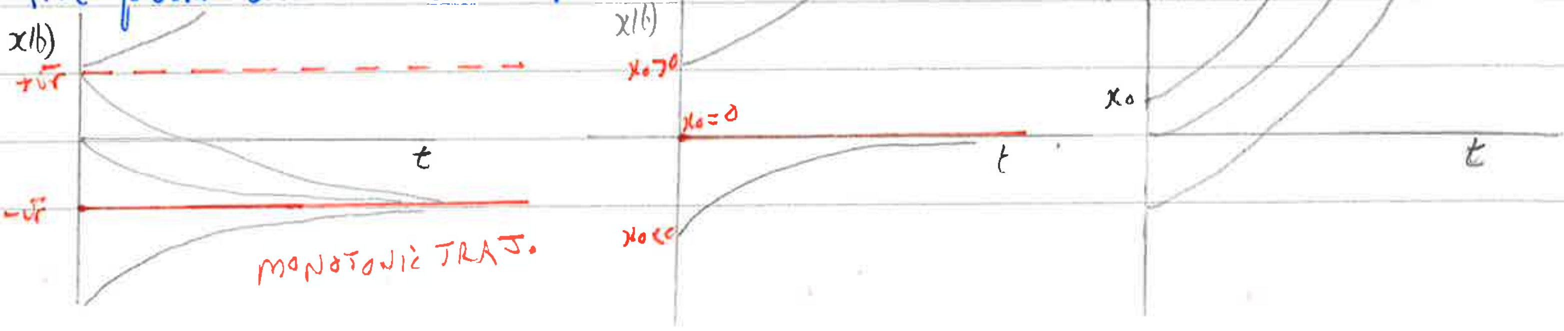
1) Saddlenode bifurcation

consider $\dot{x} = r + x^2$ $\quad r \in \mathbb{R}$ is a parameter

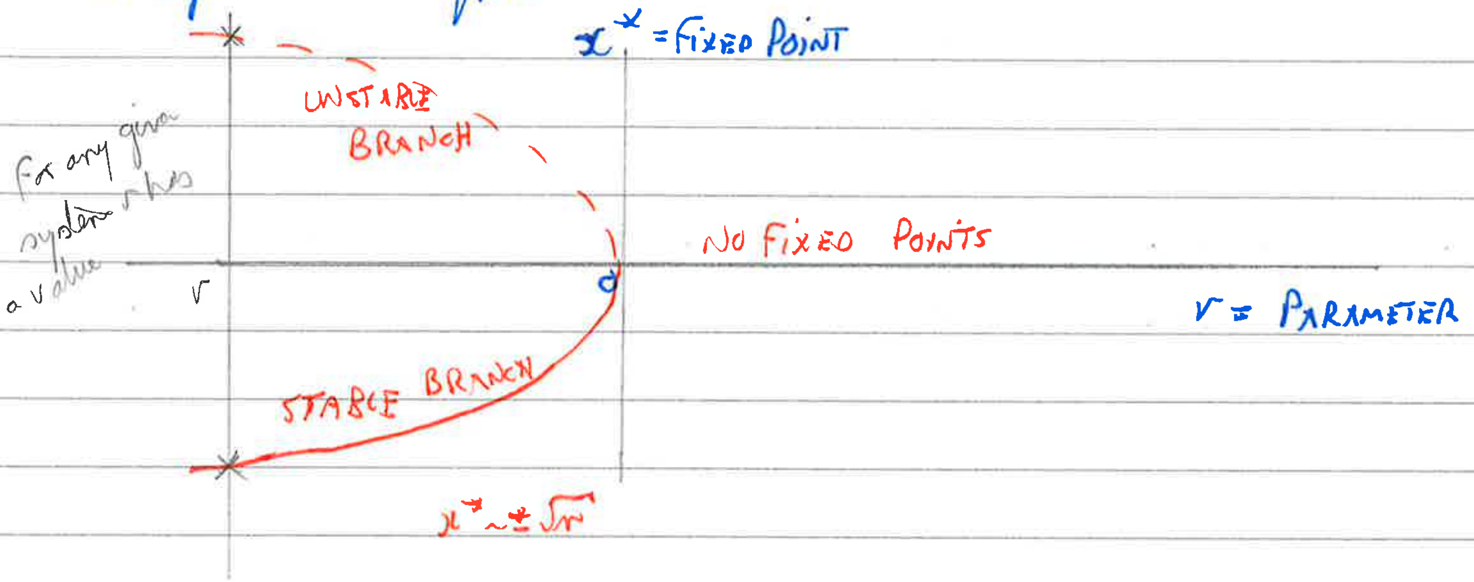
Recall Graph 1 and 2, and consider the three cases: $r<0, r=0, r>0$.



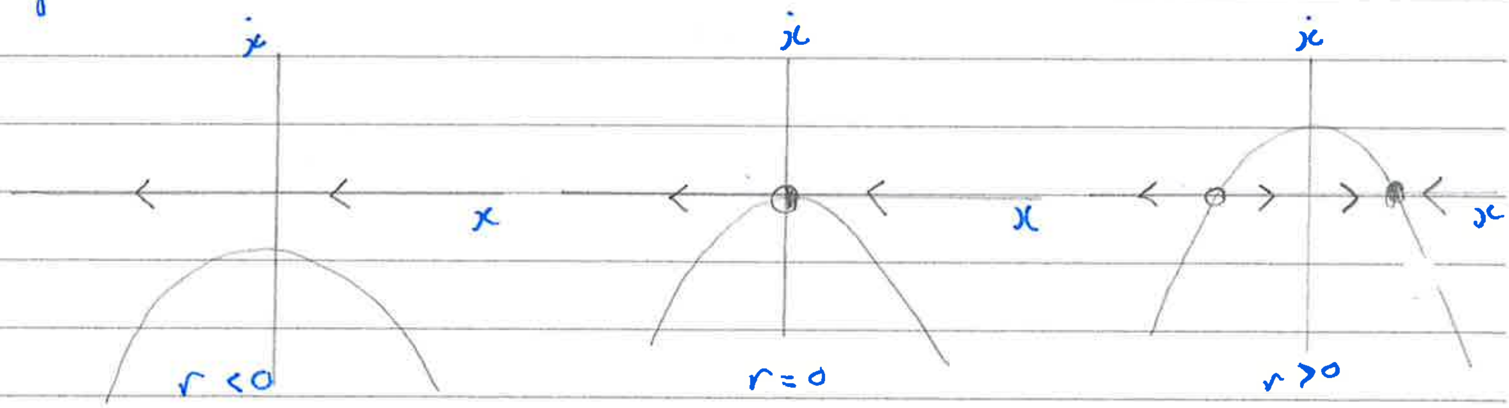
Fixed points occur at $x^* = \pm\sqrt{r}$

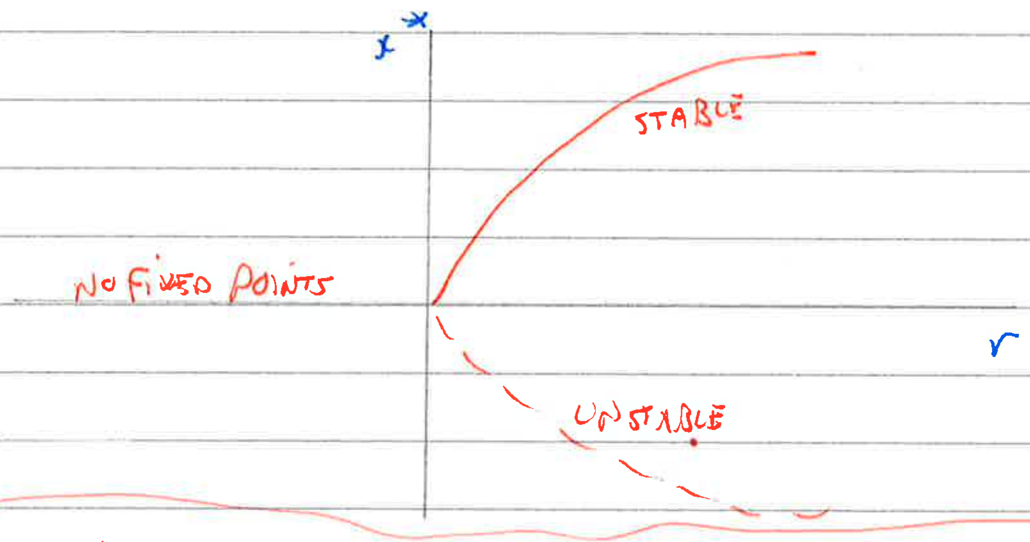


We can represent the change in the number and type of fixed points in a "Bifurcation Diagram"



EXERCISE FOR NEXT WEEK: What does the bifurcation diagram look like for $\dot{x} = r - x^2$?





2 Fixed points appear or disappear in saddlenode bifurcations as r changes

In general, we consider a 1D dynamical system that depends on a parameter r:

$$\dot{x} = f(x, r)$$

and the bifurcation diagram shows the change in the fixed points as the parameter r varies.

Question: Can fixed points appear/disappear/change in any way?

NO: There are only certain types of transitions allowed by the topology and continuity of trajectories.
NB Recall that fixed points must alternate in stability.

The graphs 1, 2 and bifurcation diagram tell us a lot about the system, which is good because even simple 1D dynamical systems have complicated solutions.

e.g. $\dot{x} = -r + x^2$

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{x^2 - r} = \int_0^t t = t \qquad \text{with solution:}$$

$$x(t) = \sqrt{r}\left[\frac{x_0(1 + e^{2\sqrt{r}t}) + \sqrt{r}(1 - e^{-2\sqrt{r}t})}{x_0(1 - e^{2\sqrt{r}t}) + \sqrt{r}(1 + e^{2\sqrt{r}t})}\right]$$

2) Transcritical Bifurcation

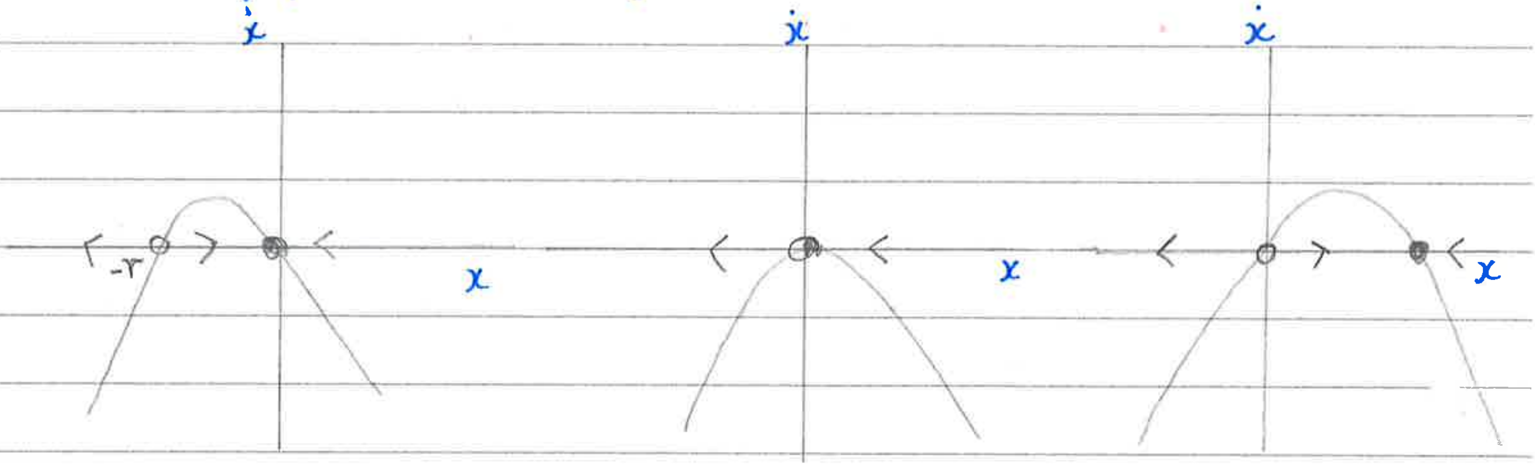
In some cases, a fixed point must exist for all values of a parameter (e.g. a population model, $\dot{N} = N \cdot R(N)$ has $N=0$ always as an F.P.) but the stability of the fixed point can change as a parameter changes.

Consider $\dot{x} = rx - x^2$ LOGISTIC MODEL

$$= x(r-x)$$

Two fixed points: $x^* = 0, r$

What does graph look like for $r<0, 0, >0$?



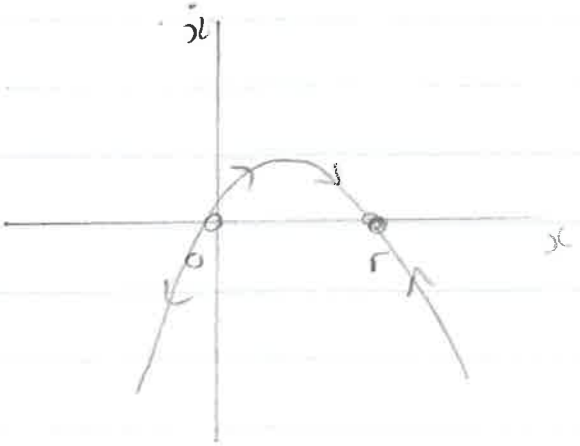
$r<0$ $r=0$ $r>0$

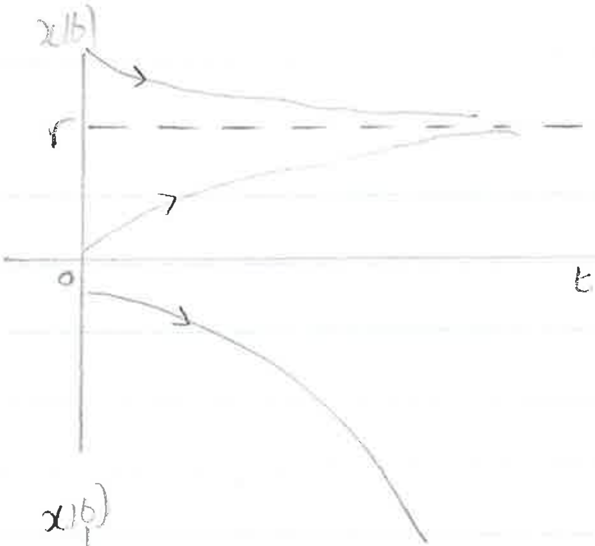
The fixed point at the origin changes stability as r varies but doesn't move.

The order of the stability of the two FPs doesn't change – the one on the right is always stable here (not always the case!)

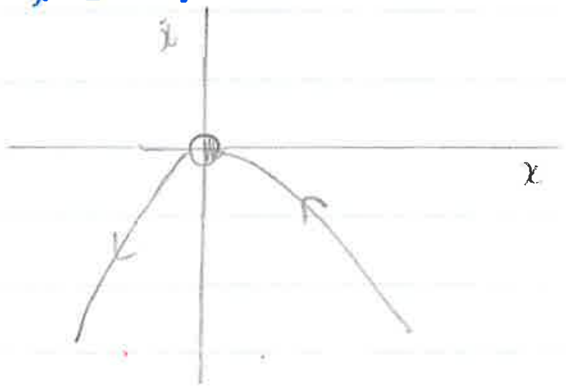
Trajectories for a Transcritical bifurcation

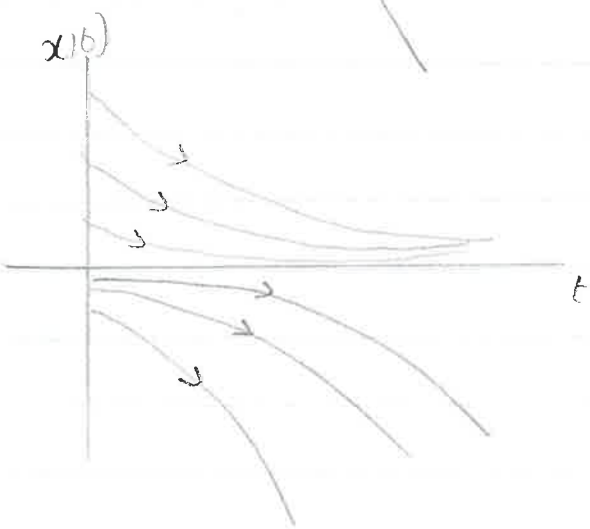
$r > 0$ $\qquad \dot{x} = rx - x^2 = x(r - x)$



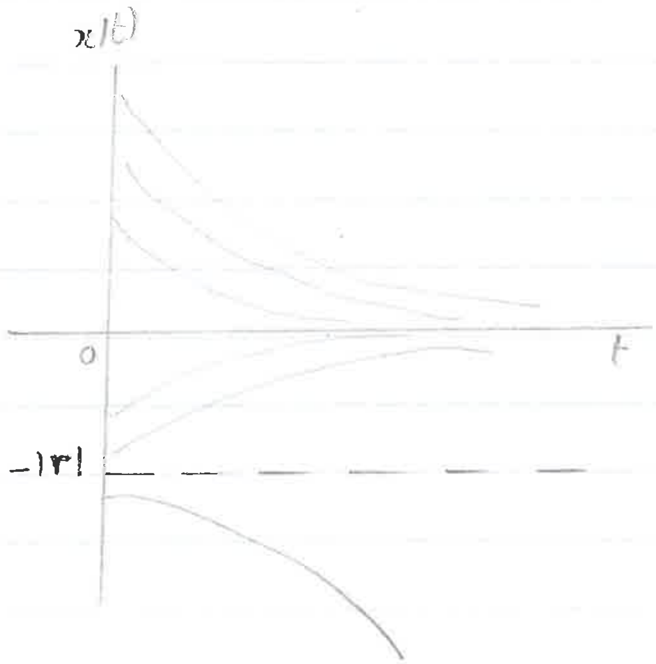


$r = 0$ $\qquad \dot{x} = -x^2$





$r < 0$ $\qquad \dot{x} = -|r|x - x^2 = -x(|r| + x)$



Let's look at the stability of the two fixed points.

$$\dot{x} = f(x, r) = rx - x^2$$

Let $\eta = x - x^*$ i.e. $\dot{\eta} = \dot{x} = f(x, r) = f(x^* + \eta, r)$

$$= \cancel{f(x^*, r)}^{\,=0} + f'(x^*, r)\cdot \eta + \cdots$$

$$= f'(x^*, r)\,\eta + \cdots$$

$$\Rightarrow \dot{\eta} = \dot{x} = f'\Big|_{x^*} = r - 2x^*$$

Case 1 $x^* = 0$: $f'(0, r) = r$

>0 if $r>0$ ∴ unstable

$=0$ if $r=0$ ∴ semi-stable

<0 if $r<0$ ∴ stable

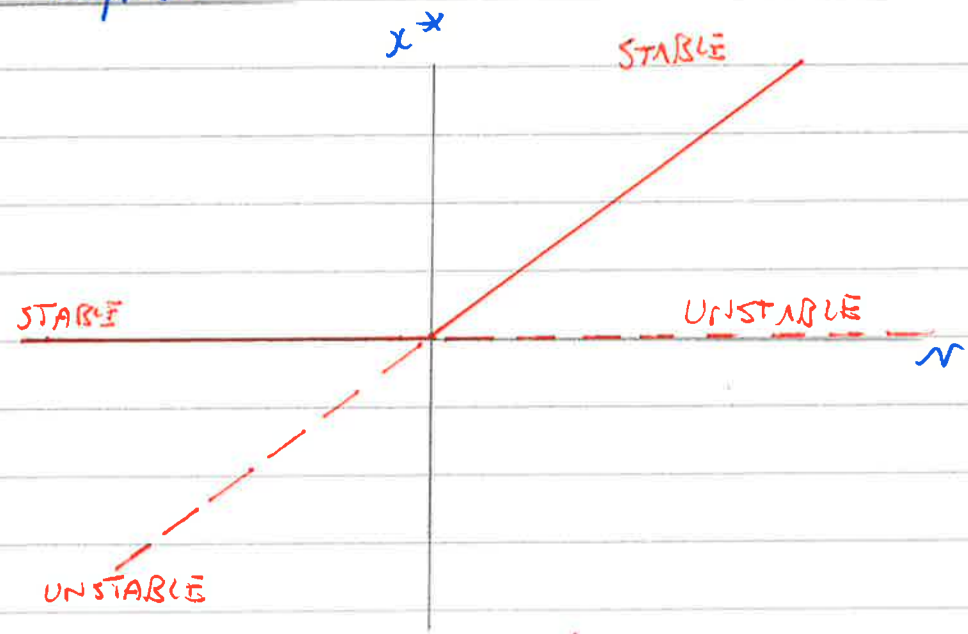
Case 2 $x^* = r$: $f'(r, r) = -r$

<0 if $r>0$ ∴ stable

$=0$ if $r=0$ ∴ semi-stable

>0 if $r<0$ ∴ unstable

Bifurcation Diagram



Here, the fixed points exchange stability as r changes

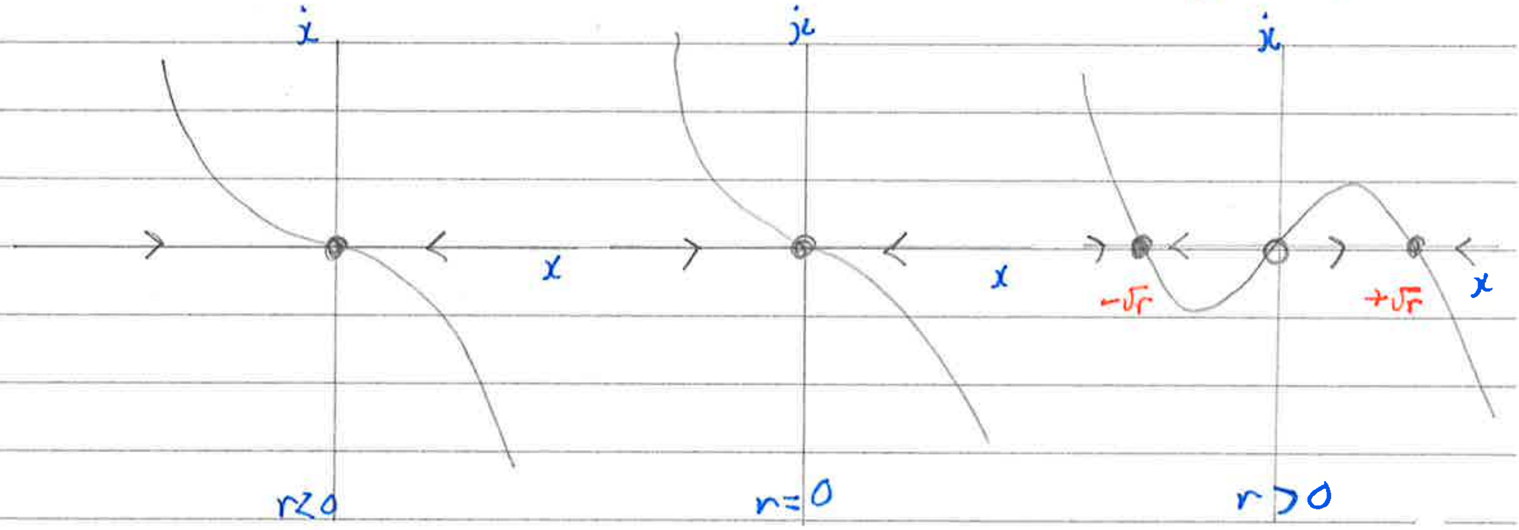
3) Pitchfork Bifurcation (the interesting one!)

A different type of bifurcation occurs in a system that has a symmetry (such as $x \to -x$), and in this case fixed points appear or disappear in pairs.

Two kinds:

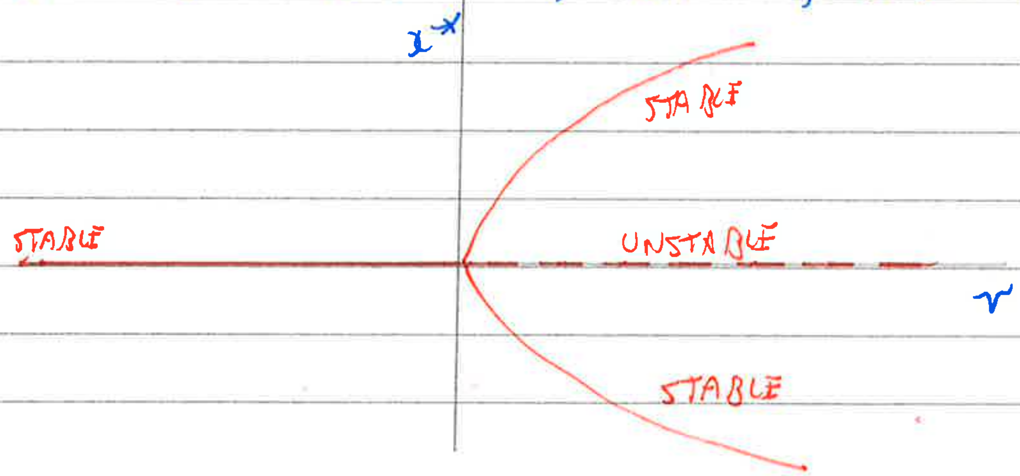
3.1) Supercritical Pitchfork bifurcation

$\dot{x} = rx - x^3$ note that $\dot{x}$ is unchanged by $x \to -x$



Where are the fixed points?

$\dot{x} = x(r - x^2) = 0 \quad \therefore x^* = 0, \pm\sqrt{r}$ if $r > 0$, otherwise $x^* = 0$.

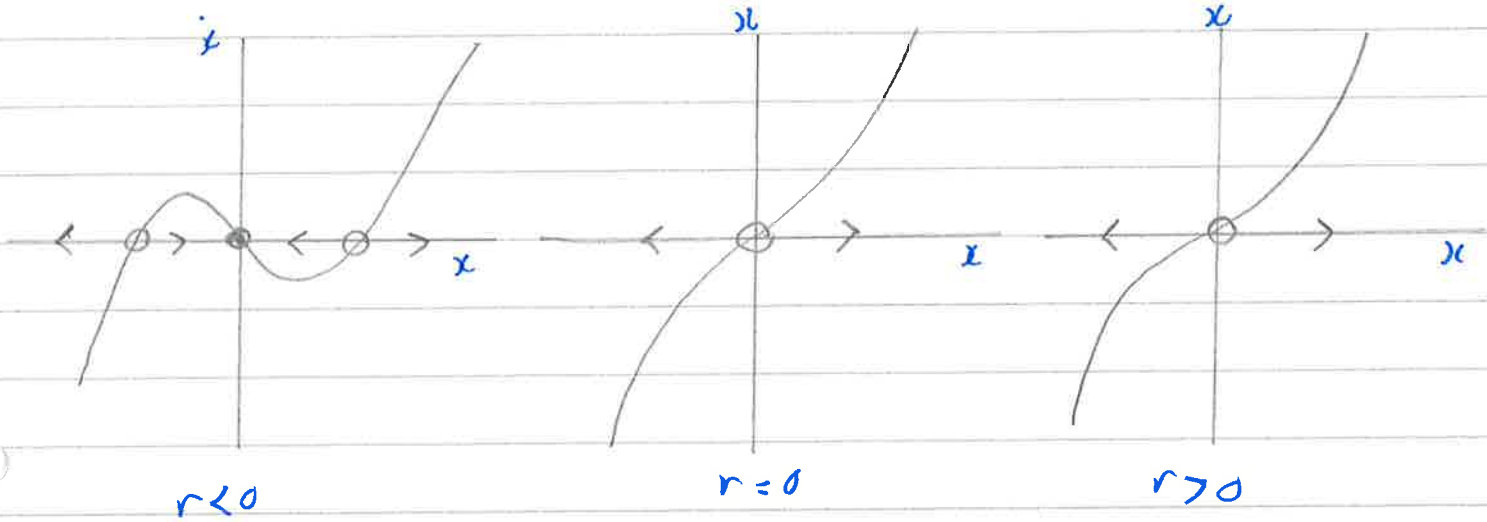


<u>Important note:</u> The stable fixed point at the origin for $r<0$ becomes unstable for $r>0$, and two new, stable fixed points appear either side of it – <u>supercritical</u> because the <u>non-zero STABLE</u> fixed points appear <u>above</u> the critical (or transition) point.

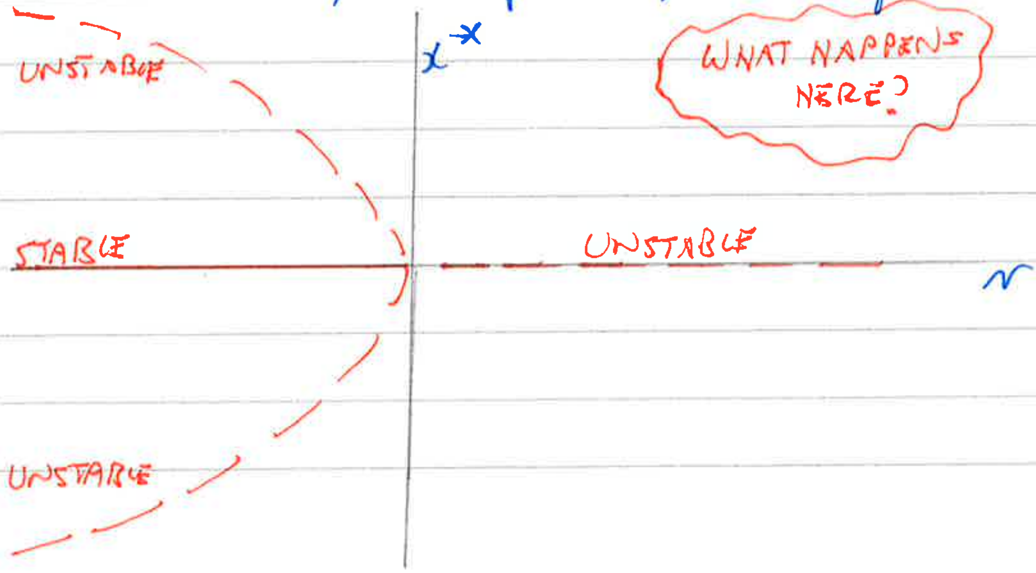
3.2) Subcritical Pitchfork Bifurcation

In the previous case, the $-x^3$ term stabilised the dynamics, i.e. it reduced $\dot{x}$, what happens if we have a destabilising term $+x^3$?

$$\dot{x} = rx + x^3 = x(r + x^2)$$



Fixed points at: $x^* = 0, \pm\sqrt{r}$ if $r<0$, otherwise just $x^* = 0$.



As in the previous case, the stable fixed point at the origin for $r<0$, becomes unstable when $r>0$, but this time there are no non-zero fixed points above the bifurcation point! So, what does a dynamical system do there?

Below the transition point there are two non-zero unstable fixed points hence SUBcritical pitchfork bifurcation.

NB. To be precise, it is the stability of the non-zero fixed point branches that determines what is a sub-critical (unstable branches) or super-critical (stable branches), not the directions they face. But the two definitions coincide here.

Bifurcations in 2D

Bifurcation also occur in 2D and higher dimensions, but typically the changes all happen in a one-dimensional subspace of the phase portrait while the flow in the other dimensions is just attraction or repulsion to/from this 1D space.

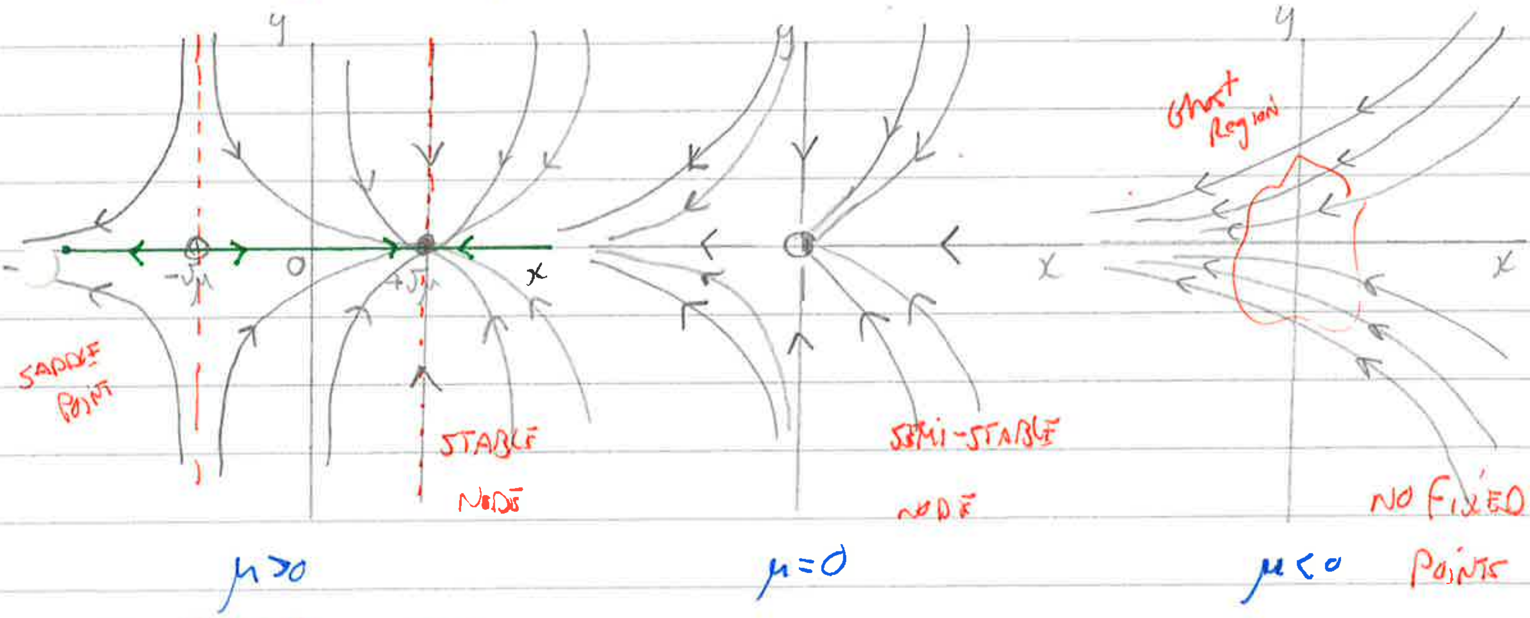
e.g. $\dot{x} = \mu - x^2$ i.e. saddlenode bifurcation in x

$\dot{y} = -y$ exponential decay of y component to zero.

What is the phase portrait?

Nullclines? $\dot{x} = 0 \Rightarrow x = \pm\sqrt{\mu}$ if $\mu > 0$

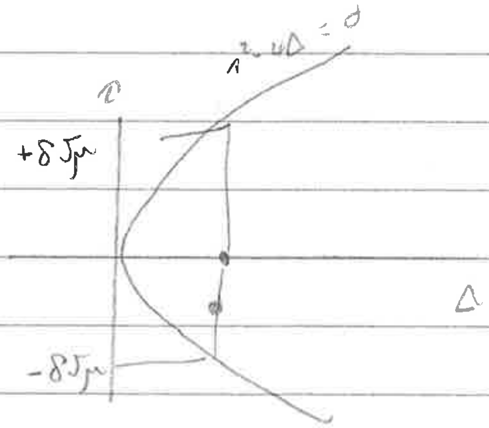
$\dot{y} = 0 \Rightarrow y = 0$



Note the different order of μ.

How do trajectories approach the stable node at $x^* = \sqrt{\mu}$? We need the eigenvectors of the Jacobian.

$$\underline{J} = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix} \quad \therefore \; \tau = -(1+2x)$$
$$\Delta = 2x$$



At $x^* = +\sqrt{\mu}$ $\Rightarrow$ $\tau = -(1+2\sqrt{\mu}) < 0$

$$\Delta = 2\sqrt{\mu}$$

$\therefore$ A stable node

because $\tau^2 - 4\Delta = (1+2\sqrt{\mu})^2 - 8\sqrt{\mu} = 1 + 4\sqrt{\mu} + 4\mu - 8\sqrt{\mu}$

$$= 1 - 4\sqrt{\mu} + 4\mu = (1-2\sqrt{\mu})^2 > 0$$

At $x^* = -\sqrt{\mu}$ $\quad \tau = -1 + 2\sqrt{\mu}$

$$\Delta = -2\sqrt{\mu}$$

$\therefore$ A saddlepoint

Eigenvalue for $+\sqrt{\mu}$

$$\begin{vmatrix} -2\sqrt{\mu} - \lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+\lambda)(2\sqrt{\mu}+\lambda) = 0$$

$$\therefore \lambda = -1, \; -2\sqrt{\mu}$$

Eiginvectors

$$\lambda = -1 \qquad \begin{pmatrix} 1-2\sqrt{\mu} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow (1-2\sqrt{\mu})\, v_1 = 0 \qquad \therefore \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

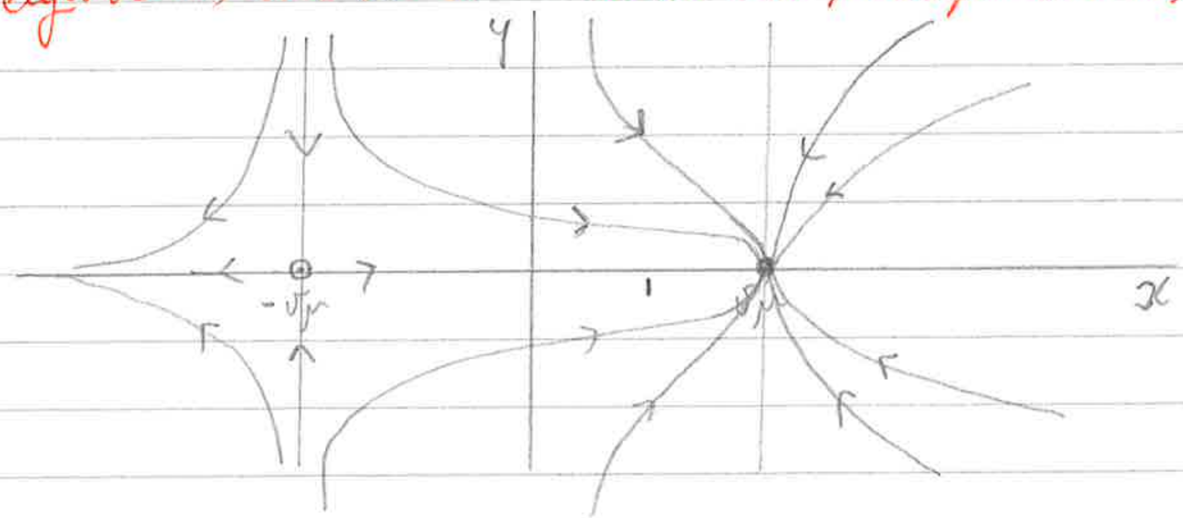
$$\lambda = -2\sqrt{\mu} \qquad \begin{pmatrix} 0 & 0 \\ 0 & -1+2\sqrt{\mu} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow (-1+2\sqrt{\mu})\, v_2 = 0 \qquad \therefore v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\therefore \underline{x}(t) = c_1 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{-2\sqrt{\mu}t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

IMPORTANT POINTS

If $2\sqrt{\mu} < 1$, the slow eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ i.e. x axis. This is the case I have drawn before. But if $2\sqrt{\mu} > 1$ then $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the slow eigenvector, and we should draw the phase portrait differently.



When $\mu < 0$, there are no fixed points. But trajectories move very slowly near $x = 0$ – a ghost region, where the fixed point used to be.

Why?

Consider the $\mu = -\varepsilon \ll 1$ i.e. close to zero but negative. How does $\dot{x}$ behave?

$$\dot{x} = \mu - x^2 \sim -\varepsilon - x^2 \ll 1 \quad \text{for } x < 1$$

As a trajectory approaches $x^* = 0$ the vector field $|\dot{x}| \ll 1$ so it slows down until it gets to $x \ll 0$ and $|\dot{x}| > 1$ again.

Note that the saddlepoint at $x^* = -\sqrt{\mu}$ for $\mu > 0$ merges with the stable node when $\mu = 0$, and they both vanish when $\mu < 0$.

Consider $\mu = 0$

$$\dot{x} = -x^2$$

$$\dot{y} = -y$$

Solve $\dot{x} = -x^2$

$$\int_{x_0}^{x} \frac{dx}{x^2} = -t$$

$$\Rightarrow \left[-\frac{1}{x} \right]_{x_0}^{x} = -t$$

$$\Rightarrow \frac{-1}{x} + \frac{1}{x_0} = -t$$

$$\Rightarrow \frac{-x_0 + x}{x x_0} = -t \quad \text{or} \quad x - x_0 = -x x_0 t$$

$$\therefore x(1 + x_0 t) = x_0 \quad \text{or} \quad \underline{x(t) = \frac{x_0}{1 + x_0 t}}$$

And we see that $x(t) \to 0$ as $t \to \infty$ if $x_0 > 0$, but if $x_0 < 0$, then $x(t)$ reaches $-\infty$ at a finite time given by $\underline{t = \frac{1}{|x_0|}}$