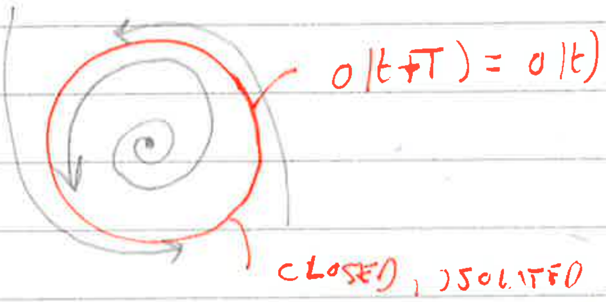
LECTURE 7: LIMIT CYCLES

(course notes: 5.1.2, Strogatz ch 7)

27/7/24

DEFINITION: A limit cycle is an ISOLATED, CLOSED trajectory.



ISOLATED ⇒ Nearby trajectories are NOT closed
They either spiral towards the limit cycle (STABLE LC)
or away from it (UNSTABLE LC)

CLOSED ⇒ A limit cycle is periodic with period T, so the variables repeat their values over time, i.e, oscillate.

NB A limit cycle is not always a circle!

NOTES

1) Limit cycles only exist in 2D or higher non-linear systems

2) Centres are closed / periodic but not isolated, i.e. nearby trajectories are closed (and also centres)

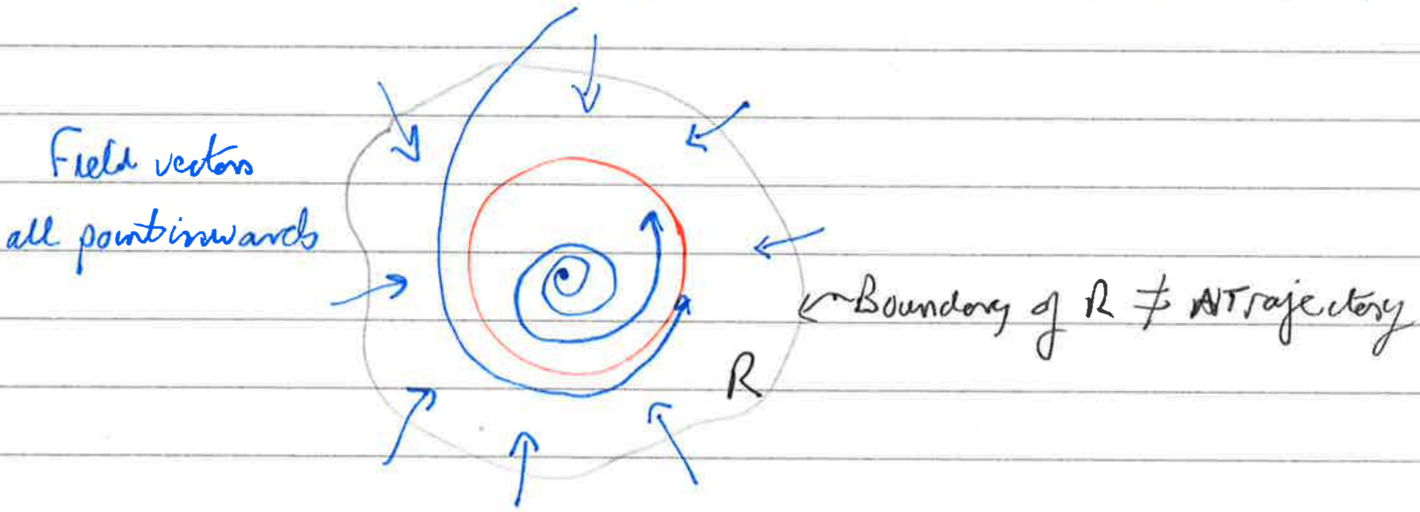
3) If you perturb a centre you move to a different trajectory, but a perturbed LC returns to the LC

∴ Limit cycles are robust

POINCARÉ - BENDIXSON THEOREM

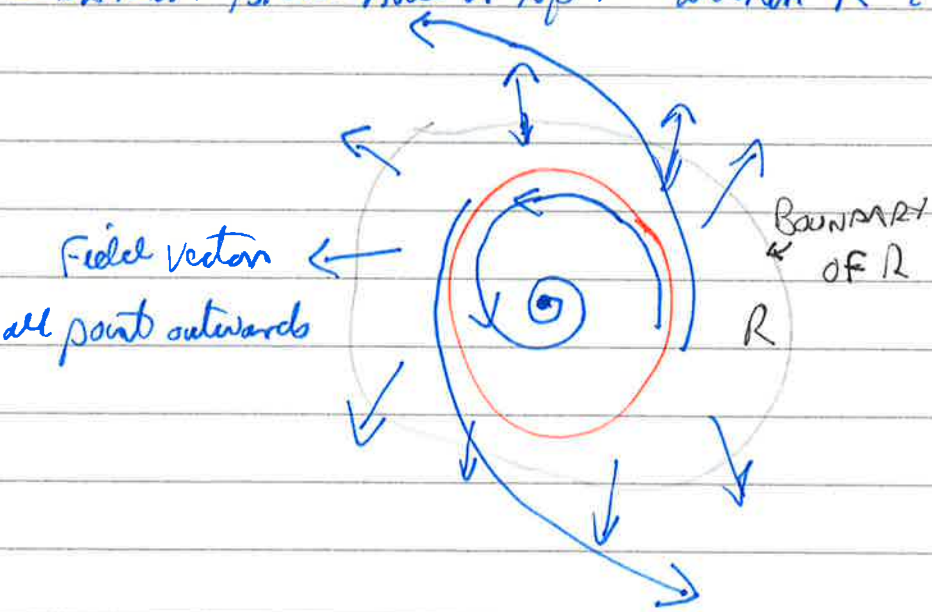
A stable limit cycle exists in a bounded region R if:

- all field vectors on the boundary of R point onto its interior
- there is an unstable node or spiral within R (NOT a saddle point)



An unstable limit cycle exists in a bounded region R if:

- all field vectors on the boundary of R point outwards
- there is a stable node or spiral within R (not a saddle point)



N.B. Semistable limit cycles also exist.

EXAMPLE (Strogatz 7.1.5)

$$\dot{x} = -y + x(1-x^2-y^2)$$

consider what the terms do:

$$\dot{y} = \underbrace{x}_{\text{Rotation}} + y\underbrace{(1-x^2-y^2)}_{\text{Forces trajectories towards } x^2+y^2=1}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

which has $Tr = 0$

$\Delta = 1$

$\therefore$ centre

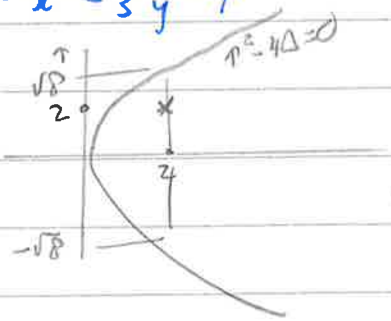
The fixed point is clearly: $x = y = 0$, What type is it?

$$\underline{\underline{J}} = \begin{pmatrix} 1 - y^2 - 3x^2 & -1 - 2xy \\ 1 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix}$$

Recipe

- Find $\underline{\underline{J}}$
- Evaluate at $(0,0)$
- classify using τ, Δ
- Find eigen value/vector

$$\Rightarrow \underline{\underline{J}}(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$



$\therefore \tau = 2$

$\Delta = 2$ $\quad \therefore$ unstable spiral at $(0,0)$

In 3D $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

Where are the nullclines? We'll come back to that!

Eigenvalues

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 + 1 = 0$$

$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$

$$\therefore \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Eigenvectors $\lambda = 1+i$

$$\begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{matrix} -iv_1 - v_2 = 0 \\ v_1 - iv_2 = 0 \end{matrix} \Rightarrow v_2 = -iv_1 \quad \therefore \underline{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$\lambda = 1-i$

$$\begin{pmatrix} 1-(1-i) & -1 \\ 1 & 1-(1-i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

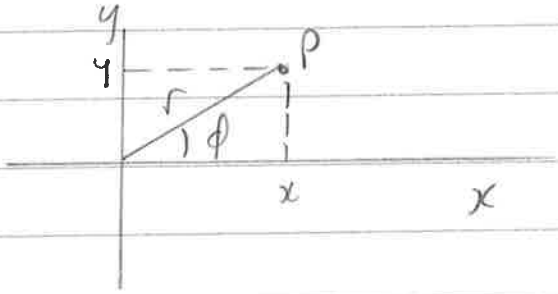
$$\Rightarrow \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{matrix} iv_1 - v_2 = 0 \\ v_1 + iv_2 = 0 \end{matrix} \Rightarrow v_2 = iv_1 \quad \therefore \underline{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\therefore$ General trajectory is: $\underline{x}(t) = c_1 e^{(1+i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(1-i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$

NB $\operatorname{Re} \lambda_i > 0 \Rightarrow$ unstable spiral.

How do we plot the trajectories? Convert to plane polar coordinates.

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \quad \begin{aligned} r^2 &= x^2+y^2 \\ \tan\phi &= \frac{y}{x} \end{aligned}$$



and $\dot{x} = -y + x(1-x^2-y^2)$

$$\dot{y} = x + y(1-x^2-y^2)$$

so, substituting for r, ϕ into these equations:

$$\boxed{\begin{aligned} \Rightarrow \dot{x} &= -r\sin\phi + r\cos\phi(1-r^2) \\ \dot{y} &= r\cos\phi + r\sin\phi(1-r^2) \end{aligned}}$$

and from $r^2 = x^2+y^2$

$$\Rightarrow \cancel{2} r\dot{r} = \cancel{2}x\dot{x} + \cancel{2}y\dot{y}$$

$$\Rightarrow r\dot{r} = r\cos\phi(-r\sin\phi + r\cos\phi(1-r^2)) + r\sin\phi(r\cos\phi + r\sin\phi(1-r^2))$$

$$= \cancel{-r^2\cos\phi\sin\phi} + r^2\cos^2\phi(1-r^2) + \cancel{r^2\sin\phi\cos\phi} + r^2\sin^2\phi(1-r^2)$$

$$= r^2(1-r^2)$$

$$\therefore \underline{\underline{\dot{r} = r(1-r^2)}}$$

and from $\tan\phi = \frac{y}{x}$

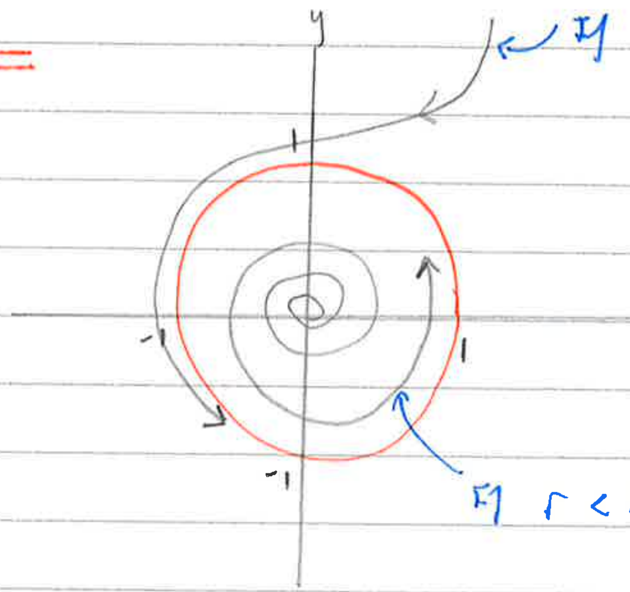
$$\Rightarrow \sec^2\phi\,\dot\phi = \frac{x\dot y - y\dot x}{x^2} = \frac{r\cos\phi\left(r\cos\phi + r\sin\phi(1-r^2)\right) - r\sin\phi\left(-r\sin\phi + r\cos\phi(1-r^2)\right)}{r^2\cos^2\phi}$$

$$= \frac{r^2\cos^2\phi + r^2\cancel{\cos\phi}\sin\phi(1-r^2) + r^2\sin^2\phi - r^2\sin\phi\cancel{\cos\phi}(1-r^2)}{r^2\cos^2\phi}$$

$$= \frac{1}{\cos^2\phi} \equiv \sec^2\phi$$

$\therefore \dot\phi = 1$

$\dot r = r(1-r^2)$



If $r > 1 \Rightarrow \dot r < 0$ spirals inwards

* Note that the nullclines are: $\dot x = 0$ at $(\pm 1, 0)$
weird curves $\dot y = 0$ at $(0, \pm 1)$
but have points

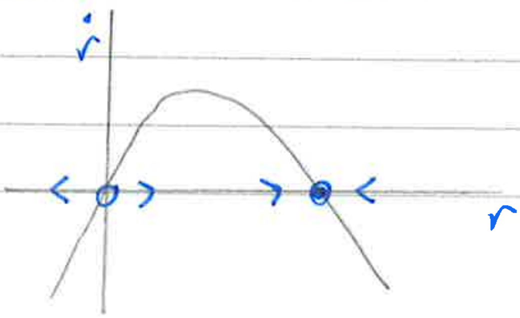
If $r < 1 \Rightarrow \dot r > 0$ spirals outwards

$\therefore$ It is a stable limit cycle.

A trapping region is any closed curve (or circle) with radius > 1.

cp Logistic Equation in 1D: $\dot x = x(1-x)$

Here we have a logistic equation in the r coordinate, while $\dot\phi = 1$ just rotates.



Selkov Model of Glycolysis
(course notes 5.2 ; Strogatz Example 7.3.2, p 207)

Glycolysis turns glucose into pyruvate, and part of this uses ATP to drive a reaction, producing ADP. It's called the glycolysis cycle, so we expect it to be periodic.

Consider $\dot{x} = -x + \underbrace{ay}_{\text{degradation}} + \underbrace{x^2y}_{\text{promotion}}$ $a, b > 0$

$\dot{y} = \underbrace{b}_{\text{source}} - \underbrace{ay}_{\text{degradation}} - \underbrace{x^2y}_{\text{suppression / competition}}$ $x = [ADP]$, $y = [ATP]$ cp Selkov p81 Ex 11 with x, y swapped.

Can we find a limit cycle in these equations?

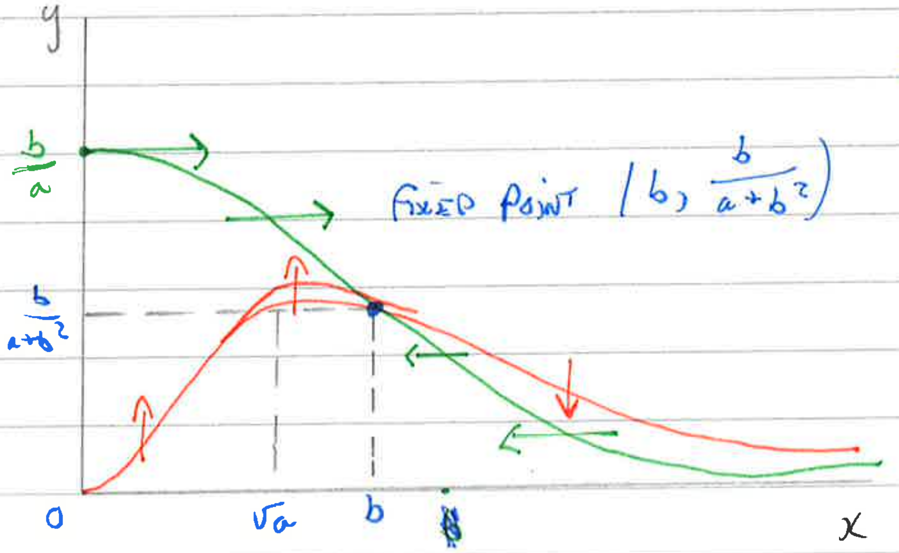
Where are the nullclines?

$\dot{x} = 0 \Rightarrow -x + ay + x^2y = 0 \quad \therefore y = \dfrac{x}{a+x^2}$ $\Rightarrow y' = \dfrac{a-x^2}{(a+x^2)^2}$

> 0 when $x = 0$

$\dot{y} = 0 \Rightarrow b - ay - x^2y = 0 \quad \therefore y = \dfrac{b}{a+x^2}$ $\Rightarrow y' = \dfrac{-2bx}{(a+x^2)^2}$

$= 0$ when $x = 0$



The fixed point is where the nullclines cross $\Rightarrow x = b$ and $y = \dfrac{b}{a+b^2}$

Note that I've made an assumption here about a, b so that the peak of the x nullcline occurs at an x value less than the fixed point, namely

$$\sqrt{a} < b \quad \text{or} \quad a < b^2$$

What is the sign of $\dot{y}$ along the x nullcline? $y = \dfrac{x}{a+x^2}$

Substituting this into $\dot{y}$ gives:

$$\dot{y} = b - ay - x^2 y = b - \cancel{(a+x^2)} \cdot \frac{x}{\cancel{a+x^2}} = b - x$$

$\therefore$ $\dot{y} > 0$ when $x < b$, and $\dot{y} < 0$ when $x > b$

and the sign of $\dot{x}$ along $\dot{y} = 0$, i.e. $y = \dfrac{b}{a+x^2}$?

$$\Rightarrow \dot{x} = -x + \cancel{(a+x^2)} \cdot \frac{b}{\cancel{a+x^2}} = b - x$$

$\therefore$ $\dot{x} > 0$ when $x < b$, and $\dot{x} < 0$ when $x > b$

How do we show there is a limit cycle in this model?

1) There is a bounded region (Trapping region) in which field vector point inwards (or outwards)

2) There is an unstable (or stable) fixed point in the trapping region.

consider 2) first. We know a fixed point is at $\left(b, \frac{b}{a+b^2}\right)$, what is its type?

$$\underline{\underline{J}} = \begin{pmatrix} -1+2xy & a+x^2 \\ -2xy & -(a+x^2) \end{pmatrix} = \begin{pmatrix} -1+\frac{2b^2}{a+b^2} & a+b^2 \\ \frac{-2b^2}{a+b^2} & -(a+b^2) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-a+b^2}{a+b^2} & a+b^2 \\ \frac{-2b^2}{a+b^2} & -(a+b^2) \end{pmatrix}$$

$$\therefore T = Tr\,\underline{\underline{J}}\left(b, \frac{b}{a+b^2}\right) = \frac{-a+b^2}{a+b^2} - (a+b^2) = \frac{-a+b^2-(a+b^2)^2}{a+b^2}$$

$$= \frac{-a+b^2-a^2-2ab^2-b^4}{(a+b^2)}$$

$$\therefore T = -\left(\frac{b^4+(2a-1)b^2+a^2+a}{(a+b^2)}\right)$$

$$\Delta = \left(\frac{-a+b^2}{\cancel{a+b^2}}\right)\cdot -(\cancel{a+b^2}) + (\cancel{a+b^2})\cdot\left(\frac{2b^2}{\cancel{a+b^2}}\right)$$

$$= a - b^2 + 2b^2$$

$$\therefore \Delta = a + b^2$$

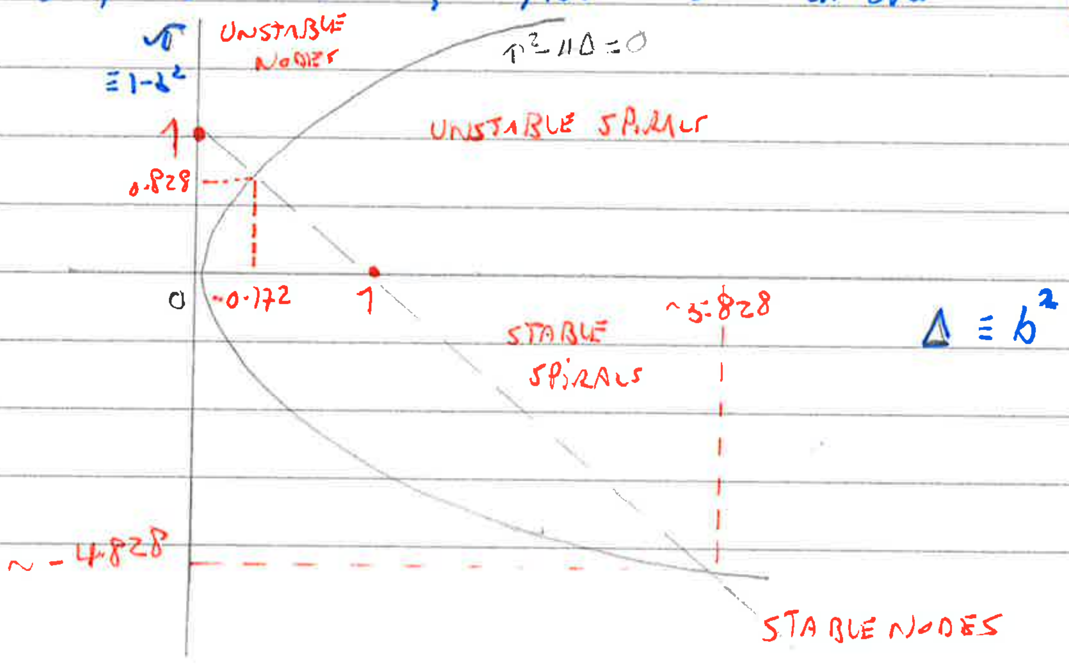
Given that $a, b > 0$, we can see that $\Delta > 0 \ \forall a, b$, so definitely not a saddlepoint. But because τ, Δ depend on two parameters a, b, it's not easy to see the type of the fixed point.

Look at a simpler case: $a = 0$

$$\therefore \tau = -\left[\frac{b^4 - b^2}{b^2}\right] = 1 - b^2$$

$$\Delta = b^2$$

As we vary b, how do τ, Δ move around in the τ-Δ plot?



Where does the straight line cross $\tau^2 - 4\Delta = 0$?

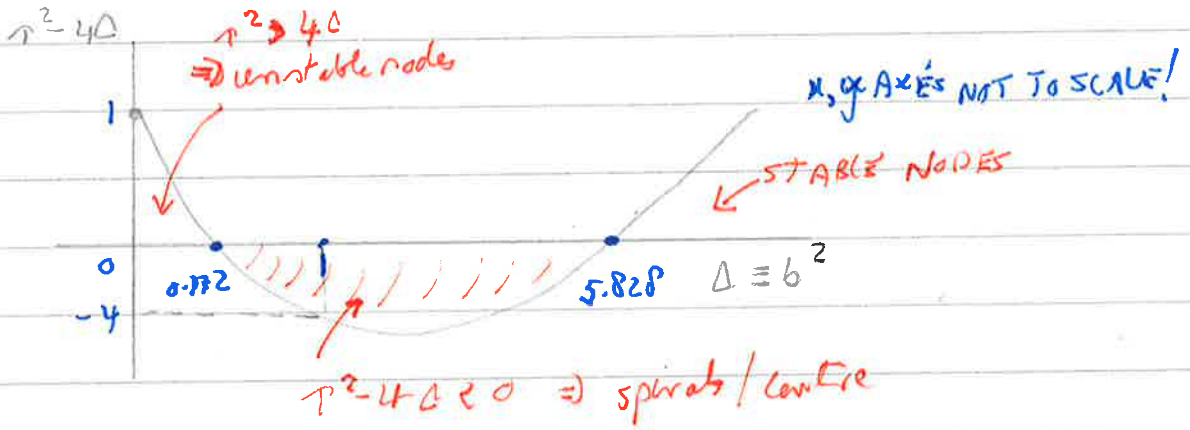
$$\tau^2 - 4\Delta = (1-b^2)^2 - 4b^2 = 1 - 2b^2 + b^4 - 4b^2$$

$$= 1 - 6b^2 + b^4 = 0$$

Consider $x = b^2$, then the equation is: $x^2 - 6x + 1 = 0$

with solutions: $x = \dfrac{6 \pm \sqrt{36-4}}{2} = \dfrac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$

$\sim 5.828,\ 0.172$



As we allow b to vary from 0 upwards, the fixed point type changes from unstable node to unstable spiral, centre, etc.

Now to the full case : $a > 0$ What is the type of the Fixed point?

consider $\tau = -\left[\dfrac{b^4 + (2a-1)b^2 + a^2 + a}{(a+b^2)}\right]$, $\Delta = a + b^2 > 0$

Let $x = b^2$ again;

$$\Rightarrow \tau = -\left[\frac{x^2 + (2a-1)x + a^2 + a}{a + x}\right]$$

$$\Delta = a + x$$

For a given value of a, we can again explore what happens to τ, Δ as we vary b. In particular - where does τ change sign? (i.e. F.P. change from unstable to stable or vice versa)

We must solve: $x^2 + (2a-1)x + a^2 + a = 0$ N.B. $a + x > 0$

$$\therefore x = \frac{1 - 2a \pm \sqrt{(2a-1)^2 - 4a(1+a)}}{2}$$

$$= \frac{1 - 2a \pm \sqrt{\cancel{4a^2} - 4a + 1 - 4a - \cancel{4a^2}}}{2}$$

$$x = \frac{1 - 2a \pm \sqrt{1 - 8a}}{2}$$

$$\Rightarrow \boxed{b^2 = \frac{1 - 2a \pm \sqrt{1 - 8a}}{2}}$$ and this must be real & positive.

$a \leq 1/8$ and when $a = 1/8$, $b^2 = \frac{1 - 2 \cdot 1/8}{2} = 3/8$ uniquely

and when $a = 0$ then $b^2 = 0$ or 1.

So, for a given value of $a \leq \frac{1}{8}$, there is a range of values of b for which $\tau > 0$, and outside this range $\tau < 0$.

e.g. $a = 0.1 \Rightarrow b^2 = \frac{1 - 0.2 \pm \sqrt{1 - 0.8}}{2} = 0.6236, 0.1764$

$\Rightarrow b = 0.79, 0.42$

b^2	τ
0	-1.1
0.1764	0
0.2	0.033...
0.5	0.067...
0.6236	0
1	-0.28

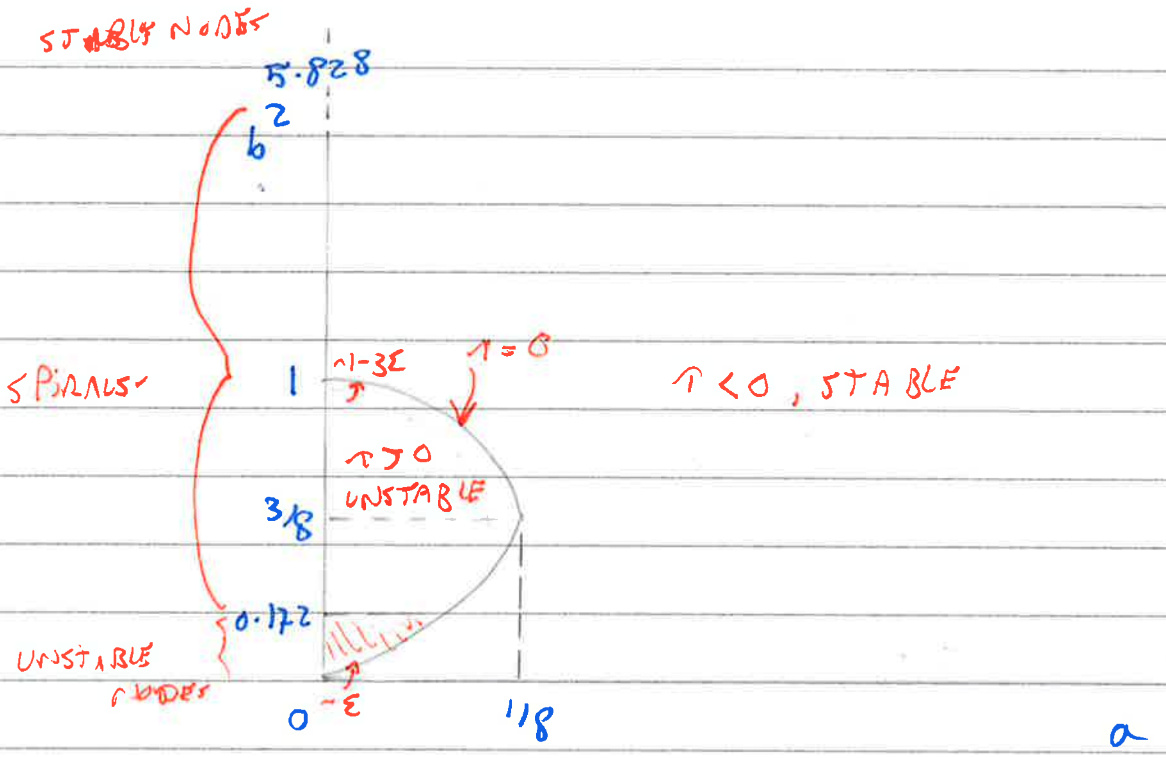
NB $b^2 = 0 \Rightarrow \tau = -(1+a)$

The bounding curve is defined by: $b^2 = \frac{1 - 2a \pm \sqrt{1 - 8a}}{2}$

Let $a = \varepsilon \approx 0$

$\Rightarrow b^2 = \frac{1 - 2\varepsilon \pm (1 - 8\varepsilon)^{\frac{1}{2}}}{2} = \frac{1 - 2\varepsilon \pm (1 - 4\varepsilon + \ldots O(\varepsilon^2))}{2}$

$= \frac{1 - 2\varepsilon \pm (1 - 4\varepsilon)}{2} = 1 - 3\varepsilon, \ \varepsilon$



For a given value of a: $0 \leq a \leq 1/8$, b^2 has a range

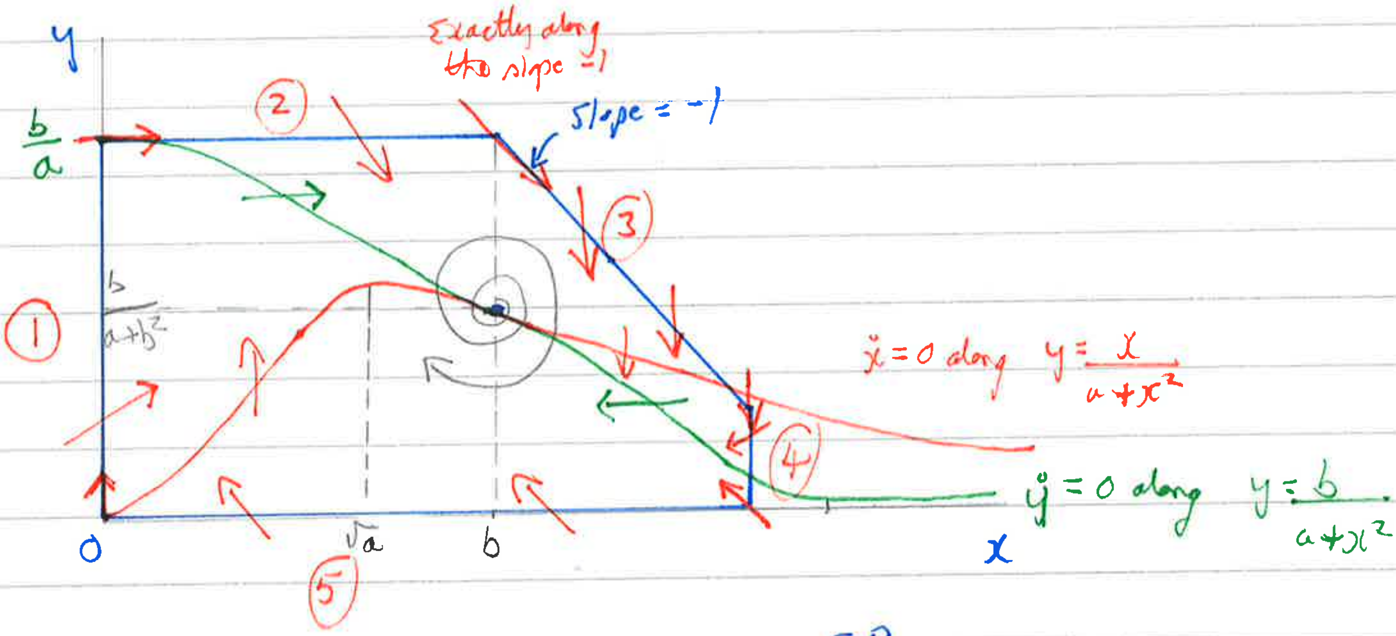
$$\frac{1-2a-\sqrt{1-8a}}{2} < b^2 < \frac{1-2a+\sqrt{1-8a}}{2}$$

What have we learned?

For a given $a < 1/8$, as we vary b we can go from a stable spiral to an unstable spiral as b^2 increases.

We now need to show there is a trapping region in order to apply the PBT, and prove the existence of a stable limit cycle.

Recall: $\dot{x} = -x + ay + x^2 y$
$\dot{y} = b - ay - x^2 y$ $a, b > 0$



Consider the 5 regions shown: which way do the field vectors point?

1) $x = 0$, $y = 0$ to b/a

$\Rightarrow \dot{x} = ay > 0$

$\dot{y} = b - ay > 0$ for $y < b/a$, and $\dot{y} = 0$ when $y = \frac{b}{a}$

and $\dot{y} = b$ when $y = 0$, $\dot{x} = 0$ when $x, y = 0$

2) $0 \leq x \leq b$, $y = b/a$

$$\dot{x} = -x + (a + x^2)\frac{b}{a} = b - x + \frac{bx^2}{a} > 0 \text{ for } x \leq b$$

$$\dot{y} = b - a\left(\frac{b}{a}\right) - x^2\frac{b}{a} = -\frac{b}{a}x^2 < 0$$

check $x = b$, $\left.\begin{aligned}\dot{x} &= \frac{b^3}{a}\\ \dot{y} &= \frac{-b^3}{a}\end{aligned}\right\} \Rightarrow \frac{dy}{dx} = -1$

4) vertical line at a value of x s.t. $x > b$ and $y=0$ up to $y = \frac{x}{a+x^2}$ The x nullcline

Recall the equation for the x nullcline: $y = \frac{x}{a+x^2}$

At the bottom when $x \geq b$, $y = 0$

$\left.\begin{aligned}\dot{x} &= -x < 0\\ \dot{y} &= b > 0\end{aligned}\right\}$ so inward to the region.

As y increases up to $y = \frac{x}{a+x^2}$, then $\dot{x} = -x + (a+x^2)y$ but $y < \frac{x}{a+x^2}$

$\Rightarrow \dot{x} < 0$ $\forall y$ up to $\frac{x}{a+x^2}$ and 0 when $y = \frac{x}{a+x^2}$

and $\dot{y} = b - (a+x^2).y$, so $\dot{y} = b$ when $y=0$ up to $b - x$ when $y = \frac{x}{a+x^2}$
which is < 0 as $x > b$.

5) This is the x axis $(y=0)$ from $x=0$ to $x>b$.

we've done the origin.

At $x = x > b$, $y=0$ $\Rightarrow$ $\dot{x} = -x < 0$, $\dot{y} = b > 0$ } true for all $0 < x \leq$ large x.

3) This is the tricky one!

This is a straight line with slope -1 from $(b, \frac{b}{a})$ to the x nullcline at $x>b$, i.e. $y = \frac{x}{a+x^2}$

consider first what are $\dot{x}, \dot{y}$ for $x, y >> 0$?

$$\left.\begin{aligned}\dot{x} &\to x^2 y\\ \dot{y} &\to -x^2 y\end{aligned}\right\} \Rightarrow \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -1$$ i.e. field vectors point downwards at 45°.

Now look at $\dot{x}, \dot{y}$ along the straight line: what is $\frac{dy}{dx}$ along here?

$$\boxed{\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b-(a+x^2)y}{-x+(a+x^2)y} = -\frac{[(a+x^2)y-b]}{[(a+x^2)y-x]}}$$

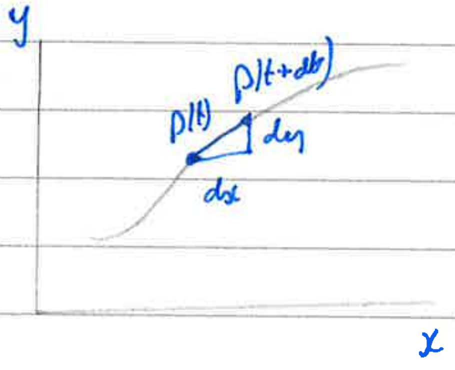
1) note that $y > \frac{x}{a+x^2}$ along the line because we are always above the curve $y = \frac{x}{a+x^2}$

so the denominator is always greater than zero.

2) $x > b$, so denominator is smaller than numerator $\Rightarrow \frac{dy}{dx} < -1$ i.e. more negative

So, the field vectors point down at an angle $\theta > 45^\circ$ i.e. inwards to the region.

Why is $y' = \frac{\dot{y}}{\dot{x}}$?



$\dot{x} = f(x, y)$
$\dot{y} = g(x, y)$

$$\left.\begin{aligned} \Rightarrow dx &\sim f\,dt \\ dy &\sim g\,dt \end{aligned}\right\} \Rightarrow \frac{dy}{dx} = \frac{g}{f} = \frac{\dot{y}}{\dot{x}}$$

Selkov model

Lecture 7