Lecture 6: Ecological Model

(Ch. 4.2 of course notes)

25/7/24 Consider a model for two interacting populations:

Logistic, competition

$$\dot N = N(1 - N - P) \equiv F(N, P)$$

$$\dot P = -P\left(\tfrac{1}{2} - N\right) \equiv G(N, P)$$

Predators increase on eating prey

PREDATORS DECLINE IF NO PREY

Because it is a population model: $N, P \geq 0$; x, y axes are nullclines; and $(0, 0)$ is a fixed point.

Fixed Points

A) $(0, 0)$

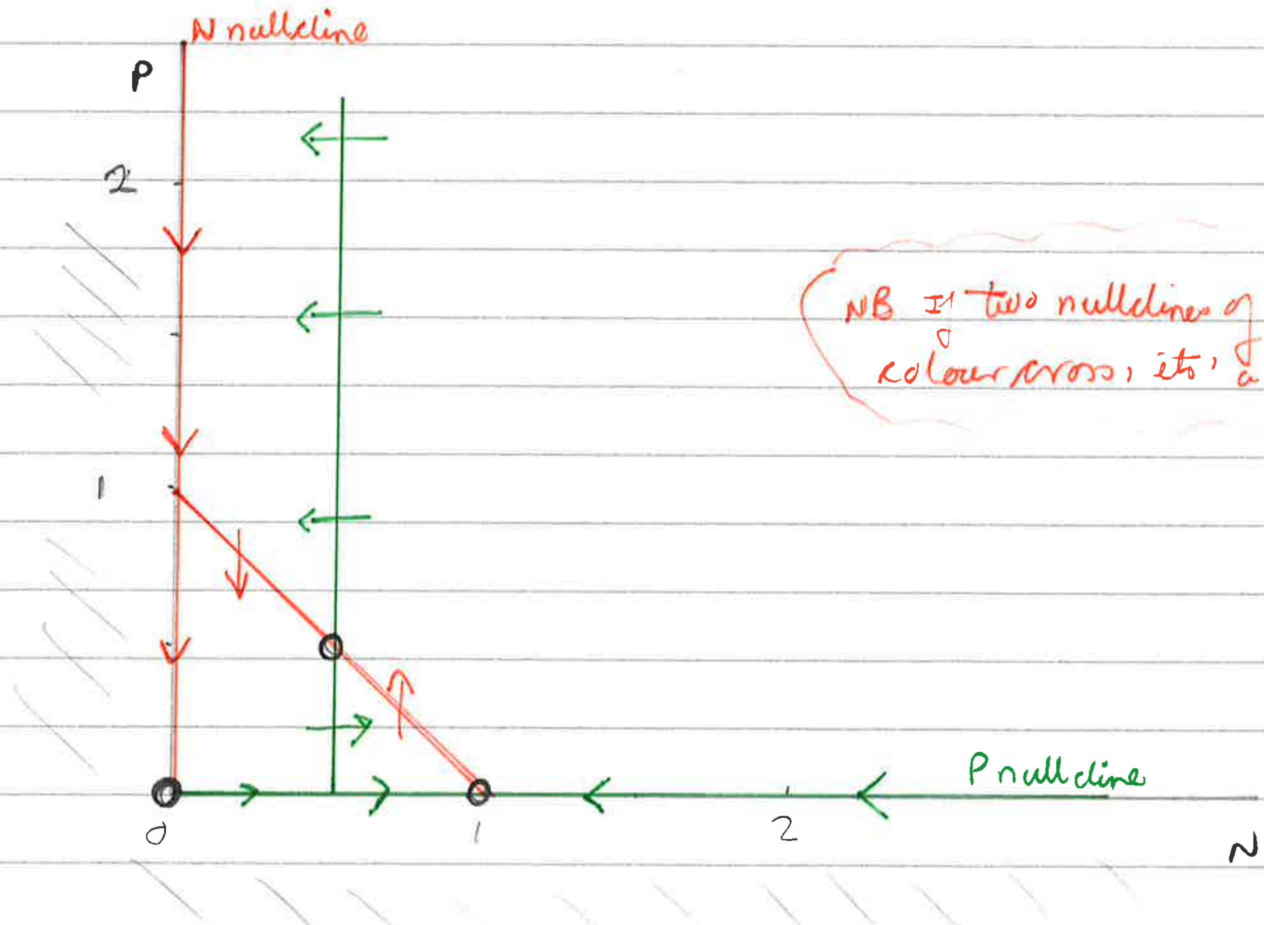
B) Try $P = 0$, then $N = 1$ $\therefore (1, 0)$

C) Try $N = \frac{1}{2}$, then $P = \frac{1}{2}$ $\therefore \left(\frac{1}{2}, \frac{1}{2}\right)$

Nullclines

$\dot N = 0 \Rightarrow N = 0$ or $P = 1 - N$

$\dot P = 0 \Rightarrow P = 0$ or $N = 1/2$



NB If two nullclines of DIFFERENT colour cross, it's a Fixed Point.

How do the vector fields look along the nullclines?

N nullcline: Along $N=0$, $\dot{P} = -\frac{1}{2}P < 0 \quad \forall P > 0$.

along $P = 1-N$, $\dot{P} = -(1-N)(\frac{1}{2}-N)$ <0 for $N < 1/2$

>0 for $\frac{1}{2} < N < 1$

<0 for $N > 1$ (unphysical)

P nullcline: Along $P=0$, $\dot{N} = N(1-N)$ >0 for $N<1$

<0 for $N>1$

Along $N = \frac{1}{2}$, $\dot{N} = \frac{1}{2}(\frac{1}{2} - P)$ >0 for $P < 1/2$

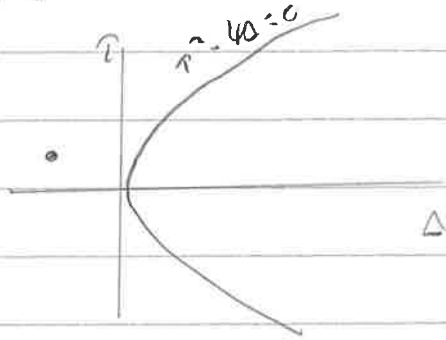
<0 for $P > 1/2$

NB The x, y axes are nullclines, so trajectories cannot cross nor leave them.

classify the fixed points

we need the Jacobian: $\underline{\underline{J}} = \begin{pmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial P} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial P} \end{pmatrix} = \begin{pmatrix} 1-P-2N & -N \\ P & N-\frac{1}{2} \end{pmatrix}$

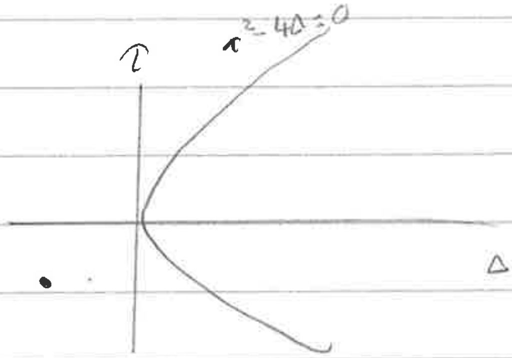
A) $(0,0)$ $\quad \underline{\underline{J}}_A = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ $\quad \therefore \tau = \frac{1}{2}$, $\Delta = -\frac{1}{2}$



$\therefore$ A saddlepoint at $(0,0)$

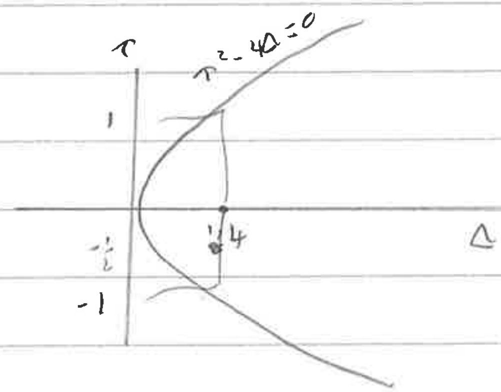
NB You can already see this from the phase portrait and nullclines.

B) $(1,0)$ $\quad \underline{\underline{J}}_B = \begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}$ $\quad \therefore \tau = -\frac{1}{2}$, $\Delta = -\frac{1}{2}$



$\therefore$ A saddlepoint at $(1,0)$

C) $(\frac{1}{2},\frac{1}{2})$ $\quad \underline{\underline{J}}_C = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ $\quad \therefore \tau = -\frac{1}{2}$, $\Delta = \frac{1}{4}$



$\therefore$ A stable spiral at $(\frac{1}{2},\frac{1}{2})$

Now we have the fixed points, what are the eigenvectors / manifolds?

A) $(0, 0)$ $\quad J_A = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ Saddlepoint

Eigenvalue: $\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\frac{1}{2}-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(-\frac{1}{2}-\lambda) = 0$

$\Rightarrow -(1-\lambda)(\frac{1}{2}+\lambda) = 0$ or $(\lambda-1)(\lambda+\frac{1}{2}) = 0$

$\therefore \lambda = -\frac{1}{2}, 1$

Eigenvectors

$\lambda = -\frac{1}{2}$ $\quad \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \frac{3}{2}v_1 = 0 \therefore \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda = 1$ $\quad \begin{pmatrix} 0 & 0 \\ 0 & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow -\frac{3}{2}v_2 = 0 \therefore \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The general trajectory is: $\underline{x}(t) = c_1 e^{-\frac{1}{2}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

STABLE MANIFOLD $\qquad$ UNSTABLE MANIFOLD

B) $(1,0)$ $\underline{\underline{J}}_B = \begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}$ Saddlepoint

Eigenvalues: $\begin{vmatrix} -1-\lambda & -1 \\ 0 & \frac{1}{2}-\lambda \end{vmatrix} = 0$

$$\Rightarrow -(1+\lambda)(\tfrac{1}{2}-\lambda) = 0$$

$$\therefore \lambda = \tfrac{1}{2}, -1$$

Eigenvectors

$\lambda = \frac{1}{2}$, $\begin{pmatrix} -3/2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow -\frac{3}{2}v_1 - v_2 = 0 \therefore v_2 = -3/2\, v_1 \therefore \underline{v}_1 = \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$

$\lambda = -1$, $\begin{pmatrix} 0 & -1 \\ 0 & -3/2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow -v_2 = 0 \therefore \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The general trajectory is: $\underline{x}(t) = c_1 e^{\frac{1}{2}t}\begin{pmatrix} 1 \\ -3/2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

UNSTABLE MANIFOLD STABLE MANIFOLD

from Strogatz, p130

The stable manifold of a saddlepoint is the set of initial points $\underline{x}_0$ that approach $\underline{x}^*$ as $t \to \infty$; and the unstable manifold is the set of initial points that approach $\underline{x}^*$ as $t \to -\infty$.

A typical trajectory asymptotically approaches the UNSTABLE manifold as $t \to \infty$, and the stable manifold as $t \to -\infty$.

c) $\left(\frac{1}{2}, \frac{1}{2}\right)$ $\underline{J}_c = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ Stable Spiral

Eigenvalues: $\begin{vmatrix} -\frac{1}{2}-\lambda & -\frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda\left(\frac{1}{2}+\lambda\right) + \frac{1}{4} = 0$

$\Rightarrow \lambda^2 + \frac{1}{2}\lambda + \frac{1}{4} = 0$

$\therefore \lambda = \dfrac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}}{2} = \dfrac{-\frac{1}{2} \pm \sqrt{-\frac{3}{4}}}{2} = -\frac{1}{4} \pm \frac{\sqrt{3}}{4} i$

$Re\, \lambda_i < 0 \Rightarrow$ stable spiral

Eigenvectors:

$\lambda = -\frac{1}{4} + \frac{\sqrt{3}}{4} i$ $\begin{pmatrix} -\frac{1}{2} - \left(-\frac{1}{4} + \frac{\sqrt{3}}{4} i\right) & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} - \frac{\sqrt{3}}{4} i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} - \frac{\sqrt{3}}{4} i & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} - \frac{\sqrt{3}}{4} i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow -\frac{1}{4}(1+\sqrt{3} i) v_1 - \frac{1}{2} v_2 = 0$ $\Rightarrow v_2 = -\frac{1}{2}(1+\sqrt{3} i) v_1$

and $\frac{1}{2} v_1 + \frac{1}{4}(1-\sqrt{3} i) v_2 = 0$

$\therefore \underline{v}_1 = \begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{3}}{2} i \end{pmatrix}$

$\lambda = -\frac{1}{4} - \frac{\sqrt{3}}{4} i$ $\begin{pmatrix} -\frac{1}{2} - \left(-\frac{1}{4} - \frac{\sqrt{3}}{4} i\right) & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} + \frac{\sqrt{3}}{4} i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} + \frac{\sqrt{3}}{4} i & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} + \frac{\sqrt{3}}{4} i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$$\Rightarrow \left(-\tfrac{1}{4}+\tfrac{\sqrt{3}}{4}i\right) v_1 - \tfrac{1}{2} v_2 = 0 \qquad \Rightarrow \quad v_2 = \left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) v_1$$

$$\tfrac{1}{2} v_1 + \left(\tfrac{1}{4}+\tfrac{\sqrt{3}}{4}i\right) v_2 = 0$$

$$\therefore \quad v_2 = \begin{pmatrix} 1 \\ -\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i \end{pmatrix}$$

The typical trajectory is: $\underline{x}(t) = c_1 e^{\left(-\frac{1}{4}+\frac{\sqrt{3}}{4}i\right)t} \begin{pmatrix} 1 \\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i \end{pmatrix}$

$$+ c_2 e^{-\left(\frac{1}{4}+\frac{\sqrt{3}}{4}i\right)t} \begin{pmatrix} 1 \\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i \end{pmatrix}$$

To complete the phase portrait, we need to know what trajectories do far from the origin, i.e. $N, P \to \infty$.

Start with $\dot{N} = N(1-N-P) \longrightarrow -N(N+P)$ for $N, P \gg 1$

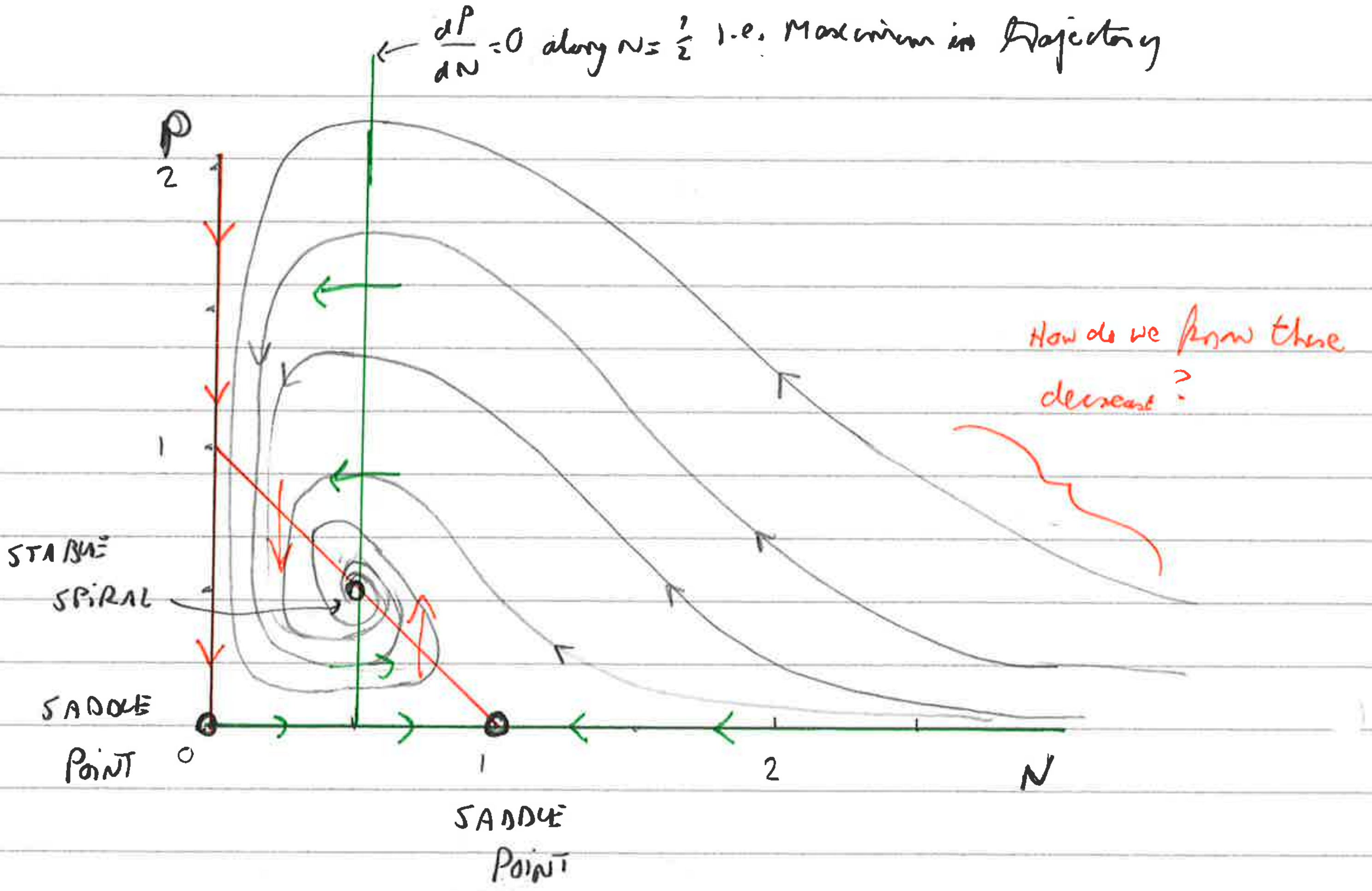
and $\dot{P} = -P\left(\tfrac{1}{2}-N\right) \longrightarrow N \cdot P$ "

$$\therefore \frac{dP}{dN} = \frac{\dot{P}}{\dot{N}} = \frac{N \cdot P}{-N(N+P)} = \frac{-P}{N+P} < 0$$

i.e. trajectories point down to the right as $N, P \to \infty$.

Note also that when $N = 1/2$, $\dot{P} = 0 \Rightarrow dP/dN = 0$ i.e. trajectories are horizontal at $N = \frac{1}{2}$

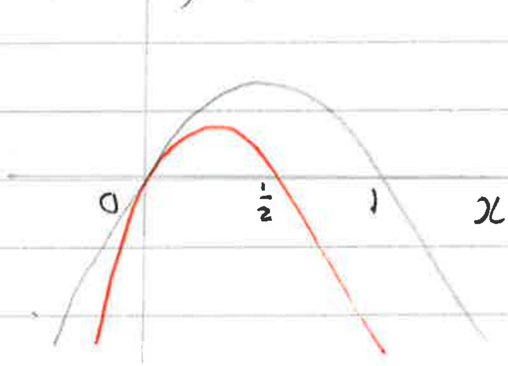
and when $N < 1/2$, and $P \to \infty$ $dP/dN > 0$ i.e. points upwards to the right



Note that all trajectories for N, P large get squeezed into $N = (0, \frac{1}{2})$.

What happens if we change the coefficients?

$\dot{x} = x(1-x)$ Logistic EQN.



$$\dot{N} = N.(1 - 2N - P)$$

smaller carrying capacity

$$\dot{P} = -P.(2 - N)$$

Faster loss of predators

Where are the fixed points now?

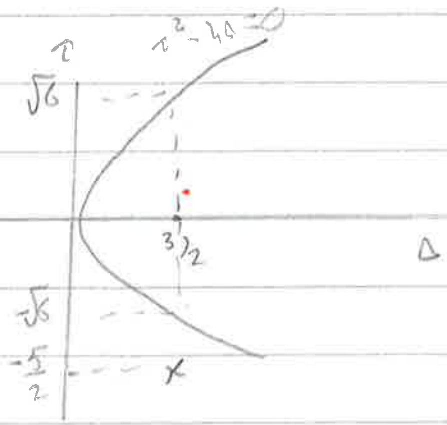
$$\underline{\underline{J}} = \begin{pmatrix} 1 - P - 4N & -N \\ P & N - 2 \end{pmatrix}$$

A) $(0,0)$ $\quad \underline{\underline{J}}_A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ $\quad \therefore \tau = -1$, $\Delta = -2$

$\therefore$ saddlepoint at the origin still

B) Try $N = 2$, which makes $\dot{P} = 0$, but then $\dot{N} = 2(-3 - P) \Rightarrow P = -3$
but this is unphysical

C) $(\frac{1}{2}, 0)$ $\quad \underline{\underline{J}}_C = \begin{pmatrix} -1 & -\frac{1}{2} \\ 0 & -3/2 \end{pmatrix}$ $\quad \therefore \tau = -5/2$, $\Delta = 3/2$



$\therefore$ stable node at $(1/2, 0)$

i.e. The saddlepoint has changed to a stable node.

Where are the nullclines?

$\dot{N} = 0 \Rightarrow N = 0$ or $p = 1 - 2N$

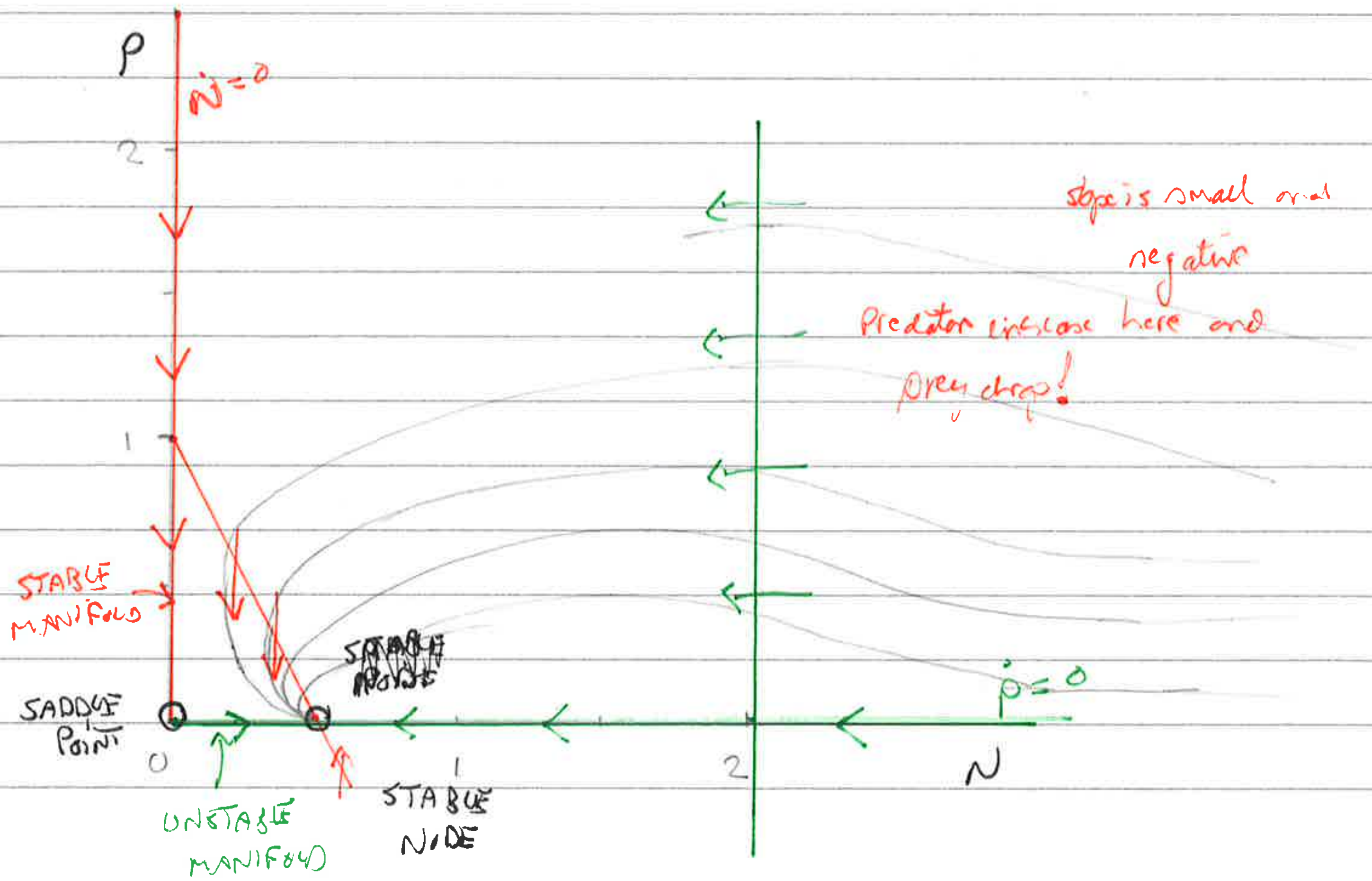
When $N = 0$, $\dot{p} = -2p < 0 \quad \forall p > 0$

and when $p = 1-2N$, $\dot{p} = -(1-2N)(2-N)$
$= (2N-1)(2-N) > 0$ for $\frac{1}{2} < N < 2$
otherwise it is negative.

$\dot{p} = 0 \Rightarrow p = 0$ or $N = 2$

when $p = 0$, $\dot{N} = N(1-2N) > 0$ for $N < 1/2$

and when $N = 2$, $\dot{N} = 2(-3-p) = -2(3+p) < 0 \quad \forall p > 0$



What happens to trajectories as $N, P \to \infty$ now?

$$\frac{dP}{dN} = \frac{\dot{P}}{\dot{N}} = \frac{-P(2-N)}{N(1-2N-P)} \rightarrow \frac{N.P}{-N(2N+P)} = \frac{-P}{2N+P}$$

similar to the original case. So, trajectories point down and right as $N, P \to \infty$.

Eigenvalues, eigenvectors of stable node at $(\frac{1}{2}, 0)$

$$\begin{vmatrix} -1-\lambda & -1/2 \\ 0 & -\frac{3}{2}-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1+\lambda)(\frac{3}{2}+\lambda) = 0$

$\therefore \lambda = -1, -3/2$

$\lambda = -1$ $\begin{pmatrix} 0 & -1/2 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_2 = 0 \quad \therefore \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = -3/2$ $\begin{pmatrix} 1/2 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \frac{1}{2} v_1 - \frac{1}{2} v_2 = 0$

$\therefore v_2 = v_1 \quad \therefore \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

slow eigenvector

FAST EIGENVECTOR

$$\therefore \underline{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-3/2\, t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$