Lecture 4: Linear 2D Dynamical Systems

(chapter 3)

The general 2D linear system is:

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

Write this as a matrix equation:

$$\dot{\underline{X}} = \underline{\underline{M}}\,\underline{X} \qquad \underline{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{\underline{M}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

NB Is this the most general case? What about $\dot{\underline{X}} = \underline{\underline{M}}\,\underline{X} + \underline{c}$ $\quad \underline{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Clearly $\underline{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a solution. Are there others?

If there are, they must satisfy: $\quad ax + by = 0$
$\qquad cx + dy = 0$

Solve for x, y from second equation to first:

$$y = -\frac{cx}{d} \quad \Rightarrow \quad ax - \frac{bcx}{d} = 0$$

$\therefore\ x(ad - bc) = 0$ so if $x \neq 0$ we must have $\det \underline{\underline{M}} = 0$.

$\therefore$ Fixed points are either: $\underline{X} = 0$ or $\det \underline{\underline{M}} = 0$.

What does $\underline{\underline{M}}$ look like if $\det \underline{\underline{M}} = 0$?

Trivial case: $\underline{\underline{M}} = 0$

Non-trivial case: $\underline{\underline{M}} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$

NB. $\underline{\underline{M}} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ reduces to trivial case $\underline{\underline{M}} = 0$ if $\underline{x} \neq 0$

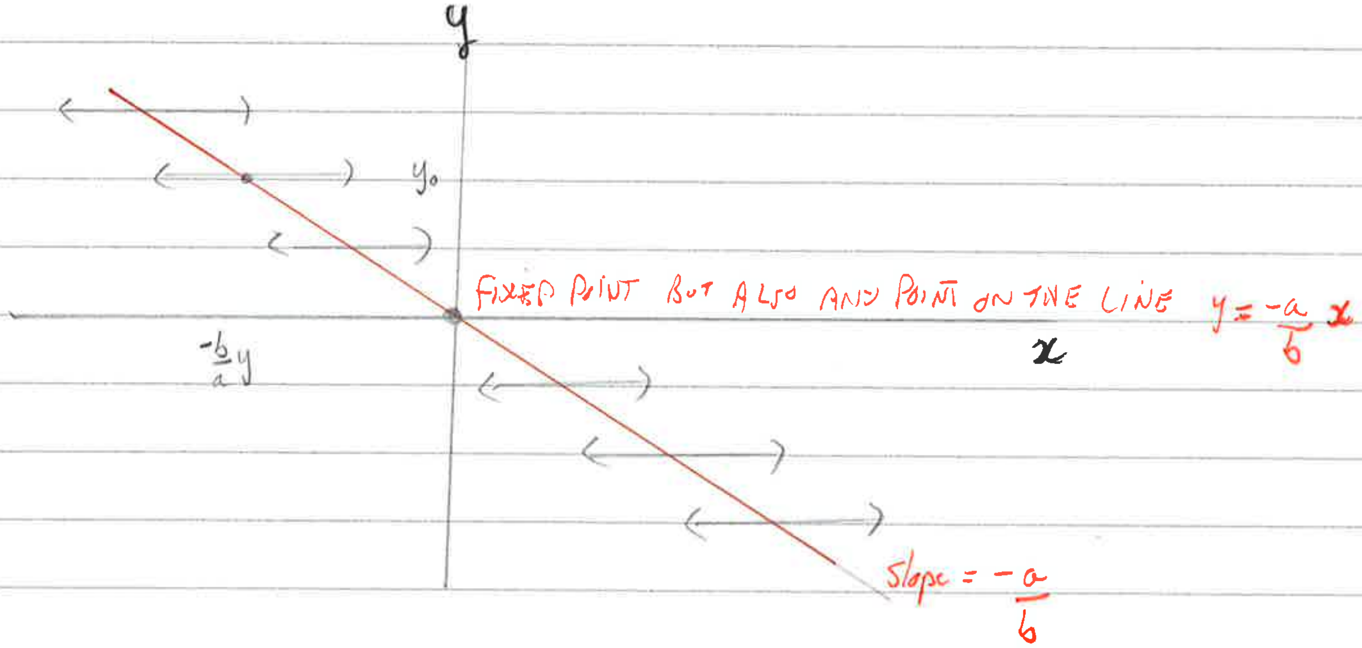
$\Rightarrow \dot{x} = ax + by$
$\dot{y} = 0$

$\therefore y(t) = y_0$ say.

$\Rightarrow \dot{x} = ax + by_0 = 0$ at a fixed point

$\therefore x^* = -\frac{b}{a} y_0$

So, the fixed point is: $(x^*, y^*) = \left(-\frac{b}{a} y_0, y_0\right)$



Shifting a 2D Fixed point to the origin

28/6/24 Suppose $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{c}$ $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ = constant

Can we transform this so that $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$?

Let $u = x + \alpha$ and find α, β to make c_1, c_2 disappear.
$v = y + \beta$

Clearly $\dot{u} = \dot{x}$ and let $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$\dot{v} = \dot{y}$

$\Rightarrow \quad \dot{u} \equiv ax + by + c_1 = a(u-\alpha) + b(v-\beta) + c_1$

$\dot{v} \equiv cx + dy + c_2 = c(u-\alpha) + d(v-\beta) + c_2$

$\Rightarrow \quad \dot{u} = au + bv - a\alpha - b\beta + c_1$

$\dot{v} = cu + dv - c\alpha - d\beta + c_2$

So, we want to find α, β so that: $-a\alpha - b\beta + c_1 = 0$

$-c\alpha - d\beta + c_2 = 0$

Solve for α: $\alpha = \dfrac{-b\beta + c_1}{a}$ and $\alpha = \dfrac{-d\beta + c_2}{c}$

$\Rightarrow \quad (-b\beta + c_1).c = (-d\beta + c_2).a$

$\Rightarrow (ad - bc)\beta = ac_2 - cc_1$

$$\boxed{\therefore \beta = \frac{(ac_2 - cc_1)}{(ad - bc)}} \qquad \text{or} \quad \beta = \frac{ac_2 - cc_1}{\det \underset{=}{A}}$$

$$\therefore \alpha = \frac{-b\beta + c_1}{a} = -\frac{b}{a}\left(\frac{ac_2 - cc_1}{(ad-bc)}\right) + \frac{c_1}{a}$$

$$= \frac{-abc_2 + b\cancel{c}c_1 + c_1(ad - \cancel{b}c)}{a(ad-bc)}$$

$$= \frac{-\cancel{a}bc_2 + \cancel{a}dc_1}{\cancel{a}(ad-bc)} = \frac{dc_1 - bc_2}{ad - bc}$$

$$\boxed{\therefore \alpha = \frac{dc_1 - bc_2}{ad - bc}} \qquad \text{or} \quad \alpha = \frac{dc_1 - bc_2}{\det \underset{=}{A}}$$

e.g. $\underset{=}{A} = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ $\quad \underset{-}{c} = \begin{pmatrix}5\\ 10\end{pmatrix}$ so $a = 1,\ b = 2,\ c = 3,\ d = 4,\ c_1 = 5,\ c_2 = 10$

$\det \underset{=}{A} = -2$ and $\alpha = \dfrac{4.5 - 2.10}{-2} = 0$

$$\beta = \frac{1.10 - 3.5}{-2} = 2.5$$

$\therefore u = x$

$v = y + 2.5$

$$\begin{pmatrix}\dot u\\ \dot v\end{pmatrix} = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}u\\ v - 2.5\end{pmatrix} + \begin{pmatrix}5\\ 10\end{pmatrix} = \begin{pmatrix}u + 2v - 5 + 5\\ 3u + 4v - 10 + 10\end{pmatrix} = \begin{pmatrix}u + 2v\\ 3u + 4v\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \quad \text{OK}$$

what if we are not on the line?

$$\left.\begin{array}{l}\dot{x} = ax + by \\ \dot{y} = 0\end{array}\right\} \quad \dot{x} = ax + by_0$$

Solve: $$\int_{x_0}^{x} \frac{dx}{ax+by_0} = \int_0^t dt$$

$$\Rightarrow \frac{1}{a}\Big[\ln(ax+by_0)\Big]_{x_0}^{x} = t$$

$$\Rightarrow \ln\left(\frac{ax+by_0}{ax_0+by_0}\right) = at$$

$$\therefore \frac{ax+by_0}{ax_0+by_0} = e^{at}$$

$$\Rightarrow ax+by_0 = (ax_0+by_0)e^{at}$$

$$\Rightarrow ax - ax_0e^{at} = -by_0 + by_0e^{at}$$

$$\Rightarrow x - x_0e^{at} = \frac{b}{a}\left(e^{at}-1\right)$$

$$\therefore x(t) = x_0e^{at} + \frac{b}{a}y_0\left(e^{at}-1\right)$$

x increases exponentially from x_0 ($a>0$, or goes to ~~[illegible]~~ $-\frac{b}{a}y_0$ if $a<0$)

note that any value of y_0 corresponds to an x^* via $x^* = \frac{-b}{a} y_0$

$$\Rightarrow x(t) = x_0 e^{at} + \frac{b}{a} y_0 (e^{at} - 1)$$

$$= x_0 e^{at} - x^* e^{at} + x^*$$

$$= (x_0 - x^*) e^{at} + x^*$$

i.e. $x(t)$ moves away from x^* exponentially proportional to its initial distance (or towards it if $a < 0$).

Nomenclature

NULLCLINE = a curve along which $\dot{x} = 0$ or $\dot{y} = 0$

FIXED POINT = intersection of the nullclines: $\dot{x} = \dot{y} = 0$

NODE : STABLE = a fixed point towards which trajectories flow as $t \to \infty$

UNSTABLE = a fixed that trajectories flow away from as $t \to +\infty$, i.e, they flow towards as $t \to -\infty$

SLOW EIGENVECTOR = The eigenvector with the smallest magnitude eigenvalue $|\lambda_i|$; trajectories approach ($\lambda < 0$) or leave ($\lambda > 0$) it along this eigenvector as $t \to \infty$

FAST EIGENVECTOR = The eigenvector with the larger magnitude; trajectories approach or leave along this eigenvector as $t \to -\infty$

SADDLE POINT = A fixed point at which trajectories approach and recede but don't touch except for its stable and unstable manifold

PTO

STABLE MANIFOLD = Set of initial points that have $x(t) \to x^*$ as $t \to \infty$

UNSTABLE MANIFOLD = Set of initial points that have $x(t) \to x^*$ as $t \to -\infty$

GENERAL CASES

(CH. 3, STROGATZ EXAMPLE 5.1.2)
COURSE NOTES

$$\underline{\underline{M}} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y \end{aligned}\right\}$$ These are uncoupled, so easy, but they illustrate the principle of different fixed points

Clearly: $x(t) = x_0 e^{at}$

$y(t) = y_0 e^{-t}$

Solutions depend on the value of the constant a.

Case 1 : $a > 0$

$$\underline{\underline{M}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

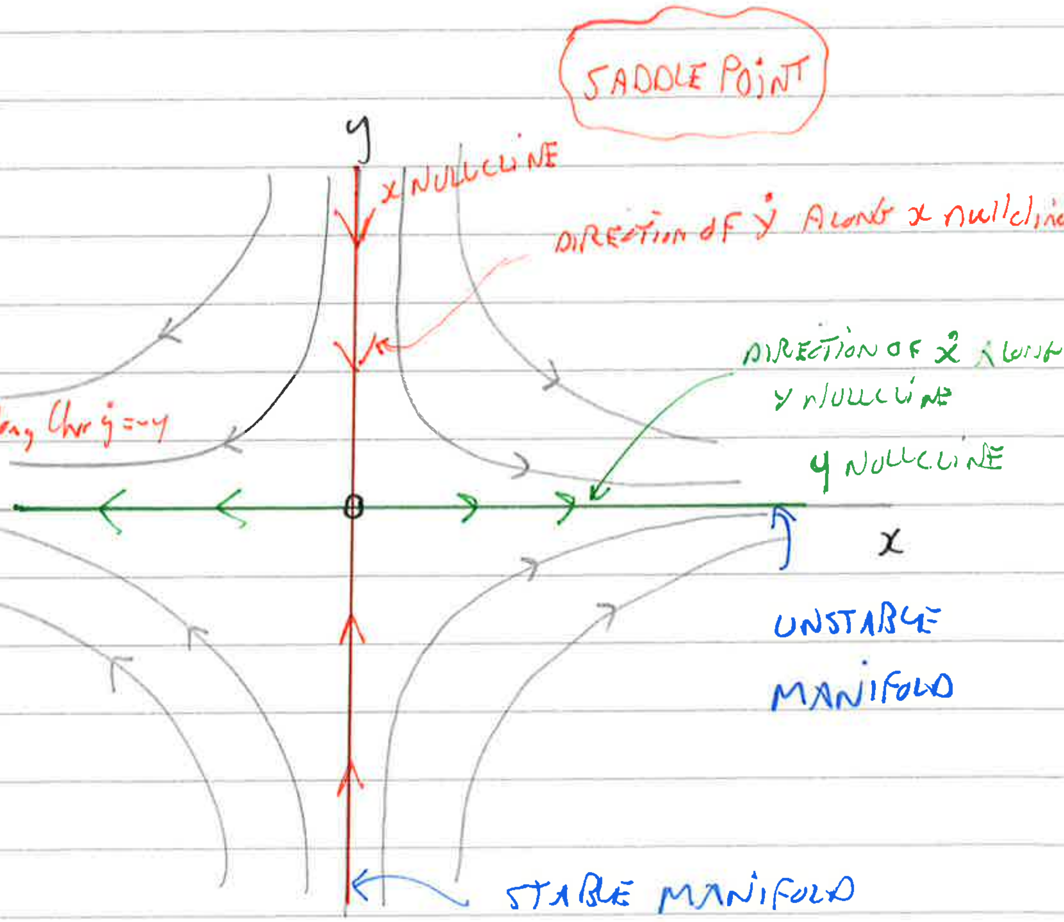
$\Rightarrow \dot{x} = x$ x nullcline is $x=0$; along this $\dot{y}=-y$

$\dot{y} = -y$ y nullcline is $y=0$

$\Rightarrow x(t) = x_0 e^{at}$

$y(t) = y_0 e^{-at}$

FP is at $(0,0)$



Case 2 $a = 0$

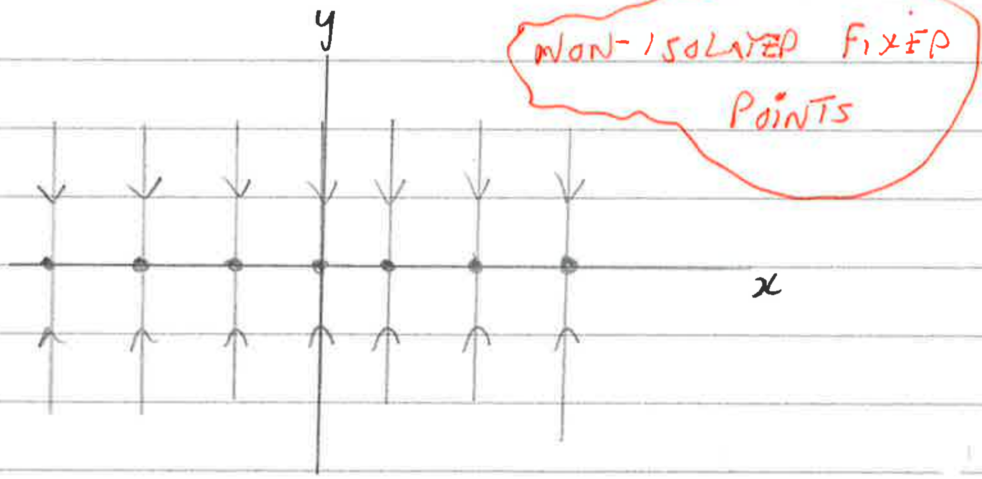
$$\underline{M} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$ NB. $\det \underline{M} = 0 \Rightarrow$ non isolated fixed point.

$\Rightarrow \dot{x} = 0$

$\dot{y} = -y$

$\Rightarrow x(t) = x_0$

$y(t) = y_0 e^{-t}$



Every value of x_0 is a fixed point. Here, they are stable because trajectories approach $y = 0$ as $t \to +\infty$.

Case 3 $-1 < a < 0$

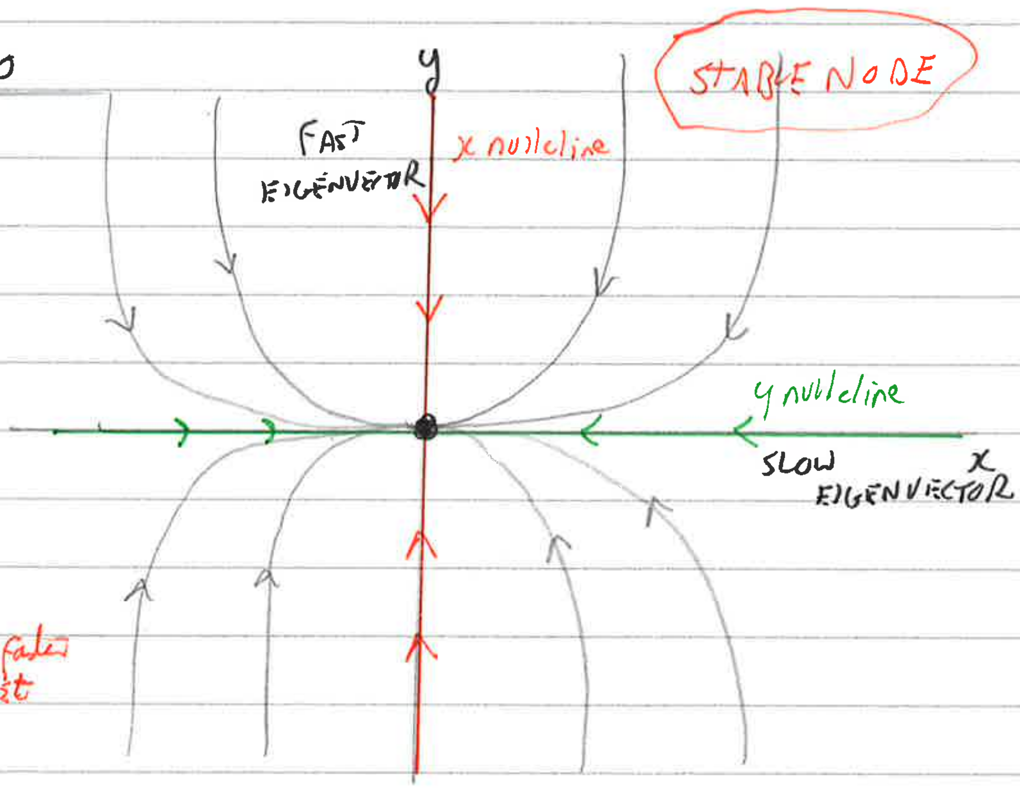
$$\underline{M} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow \dot{x} = -\frac{1}{2}x$

$\dot{y} = -y$

$\Rightarrow x(t) = x_0 e^{-\frac{1}{2}t}$

$y(t) = y_0 e^{-t}$, Decays faster than $e^{-\frac{1}{2}t}$



x nullcline is $x = 0$

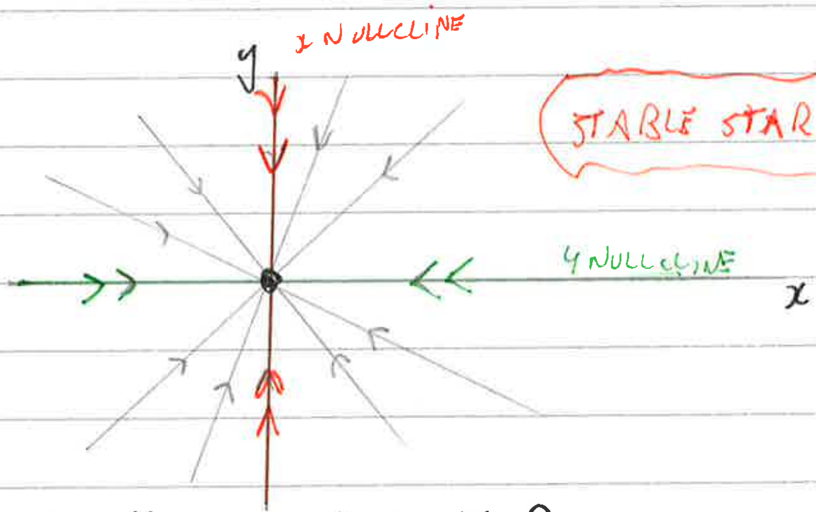
y nullcline is $y = 0$

NB The slow and fast eigenvectors are not always the same as the nullcline, nor always the axes.

Case 4 $a = -1$

STABLE STAR NODE

$$\underline{\underline{M}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$



$\Rightarrow \quad \dot{x} = -x$

$\quad\quad \dot{y} = -y$

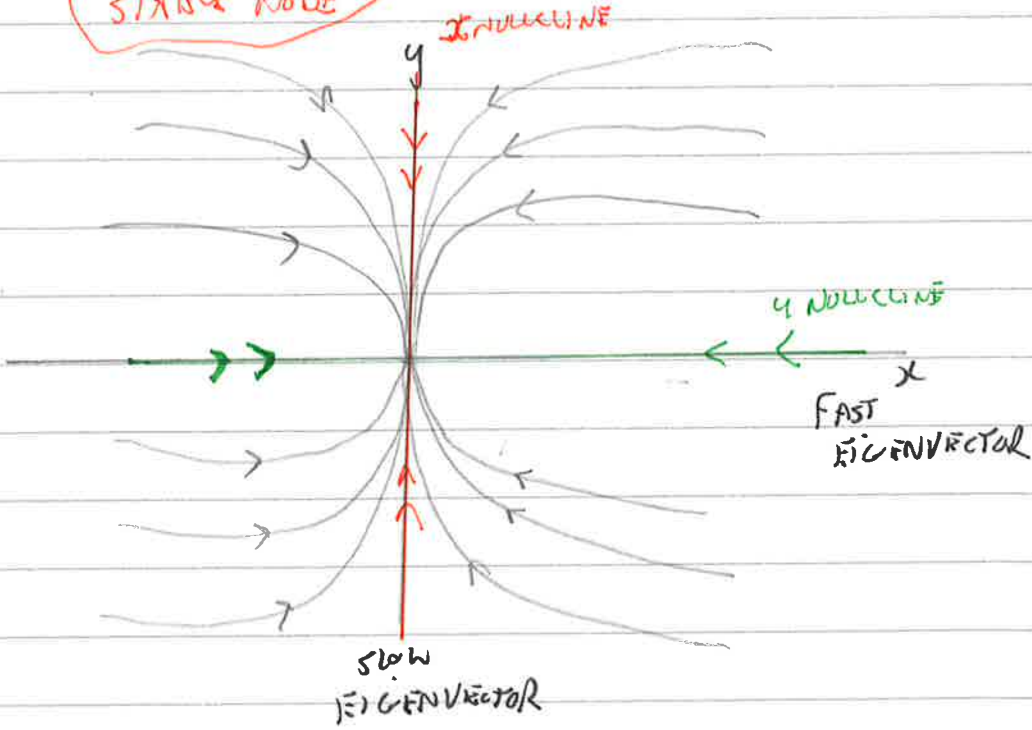
$$\Rightarrow \left.\begin{aligned} x(t) &= x_0 e^{-t} \\ y(t) &= y_0 e^{-t} \end{aligned}\right\} \Rightarrow \frac{y(t)}{x(t)} = \frac{y_0}{x_0} = \text{constant} = \tan\theta$$

Every straight line through origin is a trajectory; it is a stable star because the exponents are negative.

Case 5 $a < -1$

STABLE NODE

$$\underline{\underline{M}} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$



$\Rightarrow \quad \dot{x} = -2x$

$\quad\quad \dot{y} = -y$

$\Rightarrow \quad x(t) = x_0 e^{-2t}$

$\quad\quad y(t) = y_0 e^{-t}$

what is the general solution?

$\dot{\underline{x}} = \underline{\underline{M}}\,\underline{x}$ is a linear equation, so try the solution:

$$\boxed{\underline{x}(t) = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2}$$

General solution to linear 2D dynamical system

where λ_1, λ_2 are the eigenvalues of $\underline{\underline{M}}$
$\underline{v}_1, \underline{v}_2$ " " eigenvectors "

Then $\underline{\underline{M}}\,\underline{x} = \underline{\underline{M}}\left(c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2\right)$

$$= c_1 e^{\lambda_1 t}\underbrace{\underline{\underline{M}}\,\underline{v}_1}_{\lambda_1 \underline{v}_1} + c_2 e^{\lambda_2 t}\underbrace{\underline{\underline{M}}\,\underline{v}_2}_{\lambda_2 \underline{v}_2}$$

$$= c_1 e^{\lambda_1 t}\lambda_1 \underline{v}_1 + c_2 e^{\lambda_2 t}\lambda_2 \underline{v}_2$$

$$= \lambda_1 c_1 e^{\lambda_1 t}\underline{v}_1 + \lambda_2 c_2 e^{\lambda_2 t}\underline{v}_2$$

$\therefore \underline{\underline{M}}\,\underline{x} = \dot{\underline{x}}$

c_1, c_2 are fixed by the initial condition:

$t = 0 \Rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \underline{v}_1 + c_2 \underline{v}_2$ and we solve these simultaneous equations to get c_1, c_2 in terms of x_0, y_0.

NB If $\lambda_1, \lambda_2 > 0 \Rightarrow$ unstable node; $\lambda_1, \lambda_2 < 0 \Rightarrow$ stable node
λ_1, λ_2 have opposite signs $\Rightarrow$ saddlepoint.
λ_1, λ_2 complex $\Rightarrow$ spiral or centre fixed point.

Q. If complex, λ_1, λ_2 must be complex conjugates — why?

Recipe for 2D Linear System

1. Given the equation: $\dot{\underline{x}} = \underline{\underline{M}}\,\underline{x}$ with $\underline{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

2. find $\tau = \text{Trace } \underline{\underline{M}}$
   $\Delta = \det \underline{\underline{M}}$

3. Draw the τ-Δ plot, add the quadratic equation $\tau^2 - 4\Delta = 0$ to the rhs

4. Plot τ, Δ on the graph, and read off the type and stability of the Fixed point.
   (stop if all you need is the FP)

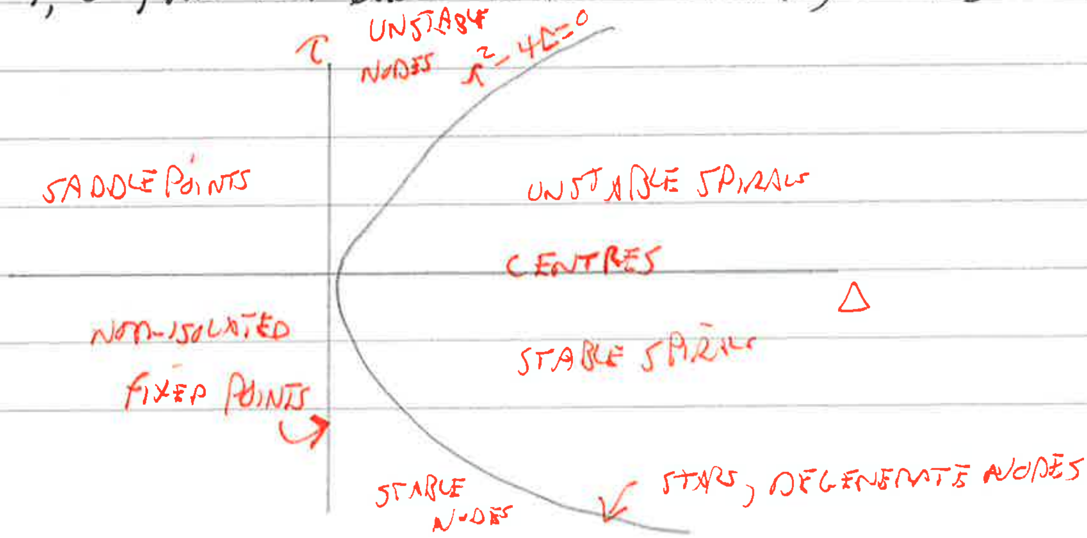
5. Find the eigenvalues, eigenvectors of $\underline{\underline{M}}$.

6. on the phase portrait, draw the nullclines $\dot{x} = 0$, $\dot{y} = 0$, note where they cross
   (Fixed point is always at 0,0 for linear system.)

7. Draw the eigenvectors on the phase portrait, and sketch the vector field.
   (stop here for qualitative solution)

8. write the general solution as: $\underline{x}(t) = c_1 e^{\lambda_1 t}\,\underline{v}_1 + c_2 e^{\lambda_2 t}\,\underline{v}_2$

   and find c_1, c_2 from the initial condition $(x_0, y_0) = c_1 \underline{v}_1 + c_2 \underline{v}_2$

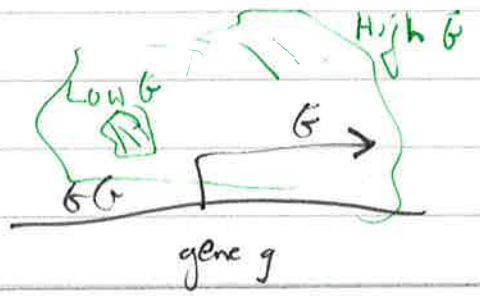


Blackboard Drawings

(Strib ch 2)

SIDE

Where are we going? $\frac{dg}{dt} = s - rg + \frac{g^2}{1+g^2}$



How can we create a switch, robust against small changes in [G]?

MAIN

Fixed points are the key : Let $f(g) = rg - s$

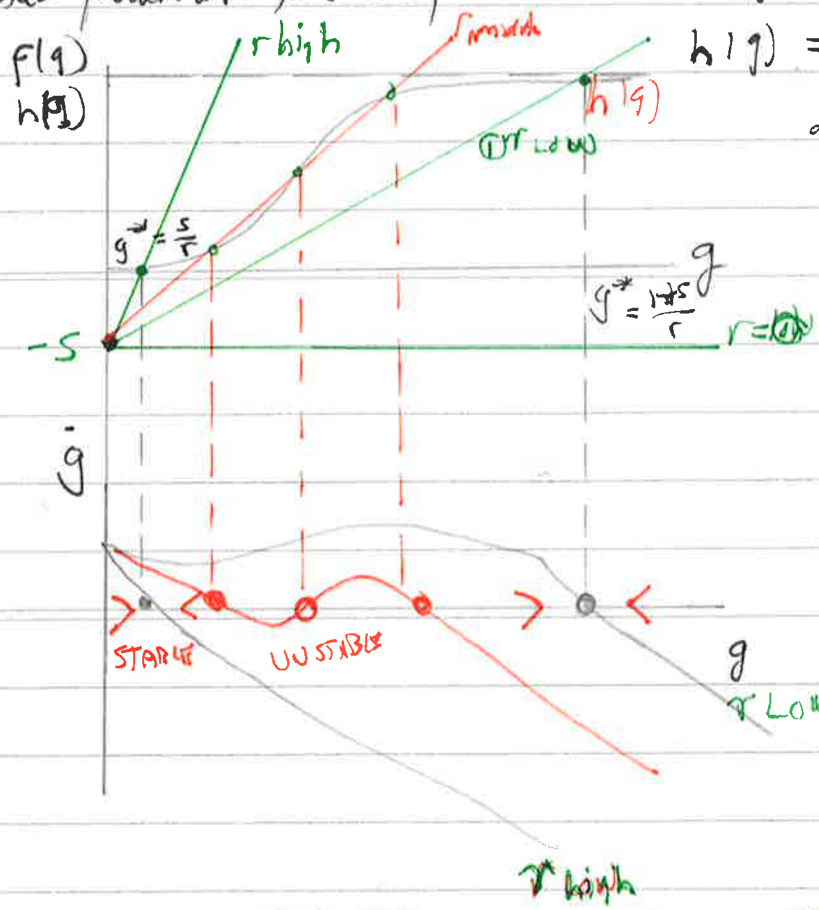
$h(g) = g^2/(1+g^2)$

and look for $f(g^*) = h(g^*)$

sign of $\dot{g}$ determined by $h(g) - f(g)$

i.e. curve − line

GRAPH 1

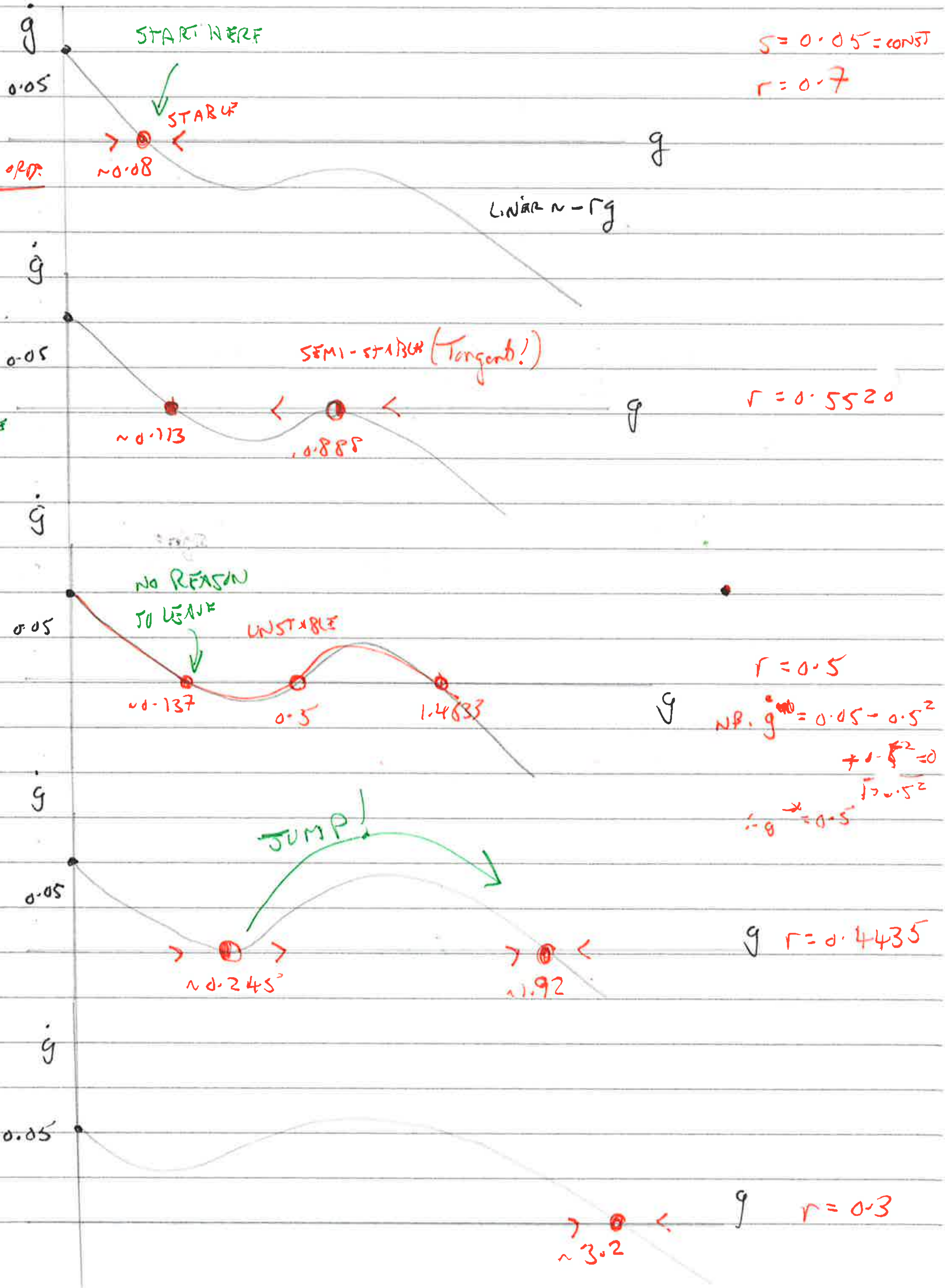


An unstable FP appears between FPs at low and high [G] preventing a continuous change.

Sequence of Graph 1 for varying r: $\dot{g} = s - rg + \frac{g^2}{1+g^2}$

DRAWING ORDER:
- AXES
- S small
- r small
- TOP DOWN

- NO REASON TO LEAVE
- JUMP



Tangent Conditions

Q For what pairs of (r, s) does the unstable FP first appear?

graphically & algebraically, we require $\dot{g} = 0$ i.e. $f(g) = h(g)$

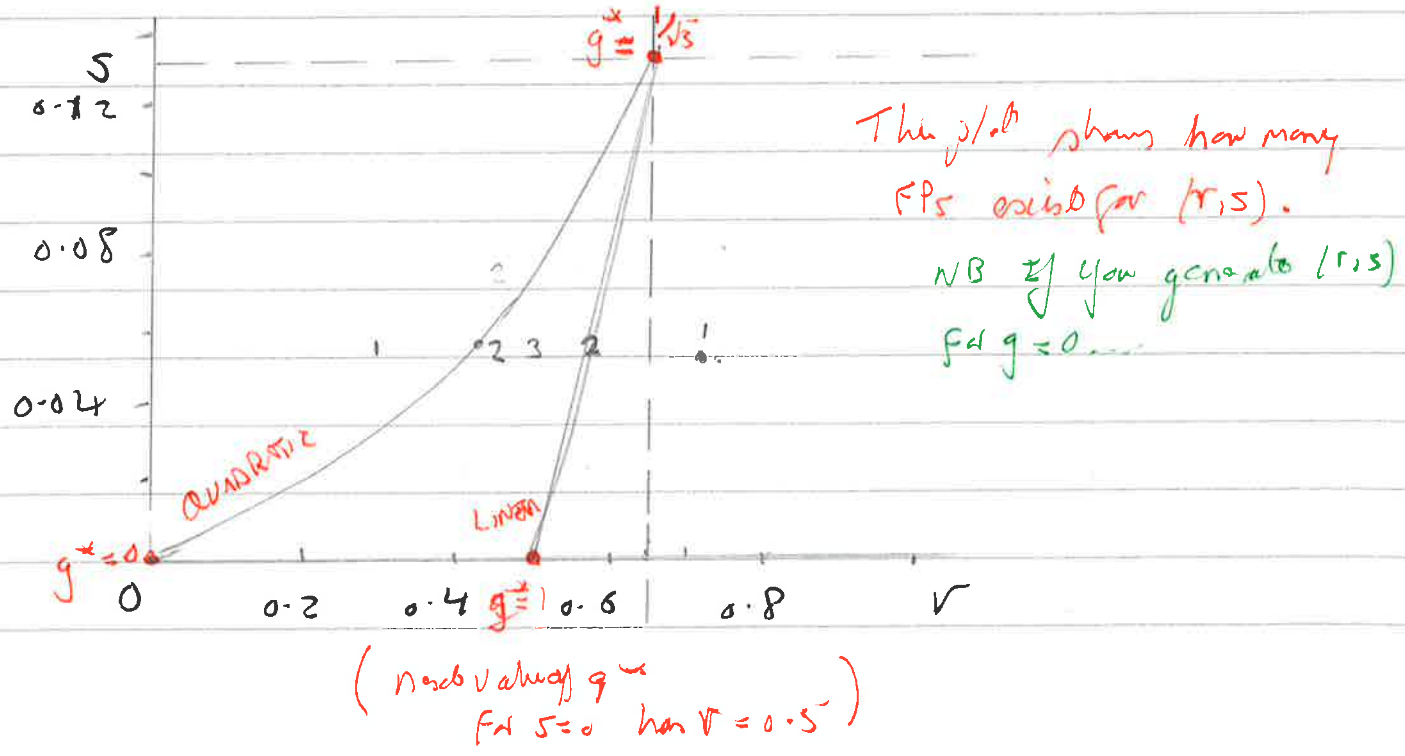
$$\frac{d\dot{g}}{dg} = 0 \quad \text{i.e. } f'(g) = h'(g)$$

$$\Rightarrow \quad r = \frac{2g}{(1+g^2)^2}, \quad s = \frac{g^2(1-g^2)}{(1+g^2)^2}$$

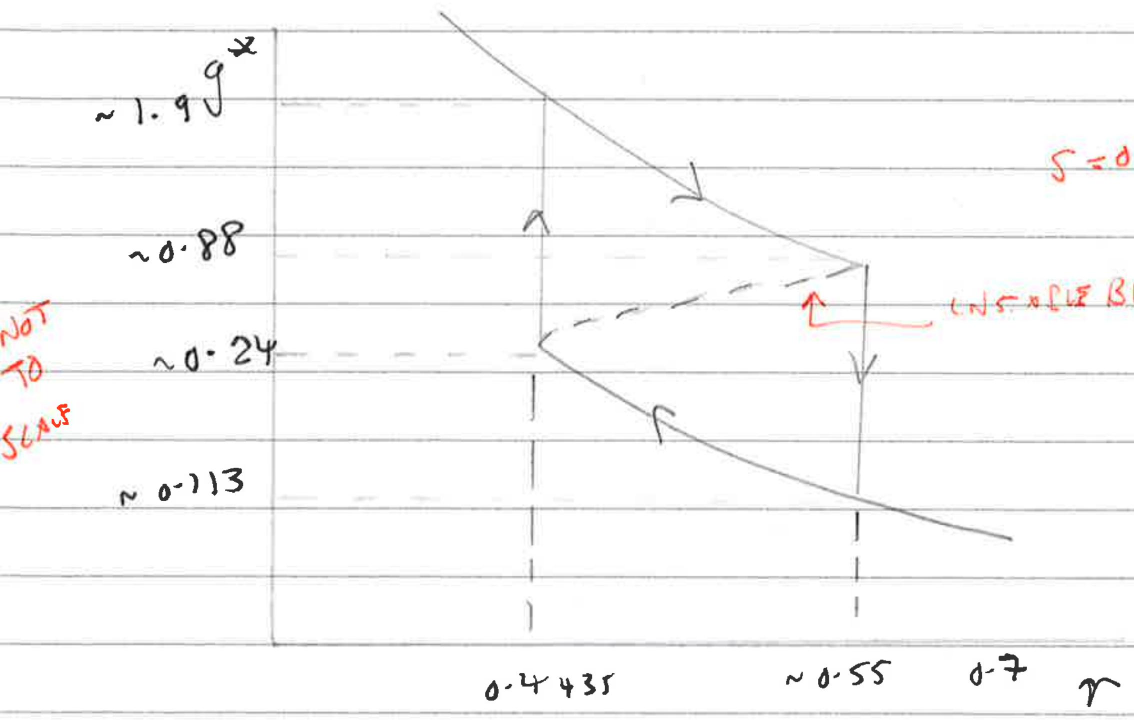
These are implicit equations: we would like $g(r, s)$ but we have $r(g), s(g)$. Notice that r, s have maximum values that can satisfy these equations because rhs goes to zero as $g \to \infty$.

$$\frac{dr}{dg} = 0 \Rightarrow g^{*2} = 1/3 \Rightarrow r_{max} = \frac{9}{8\sqrt{3}} \sim 0.65$$

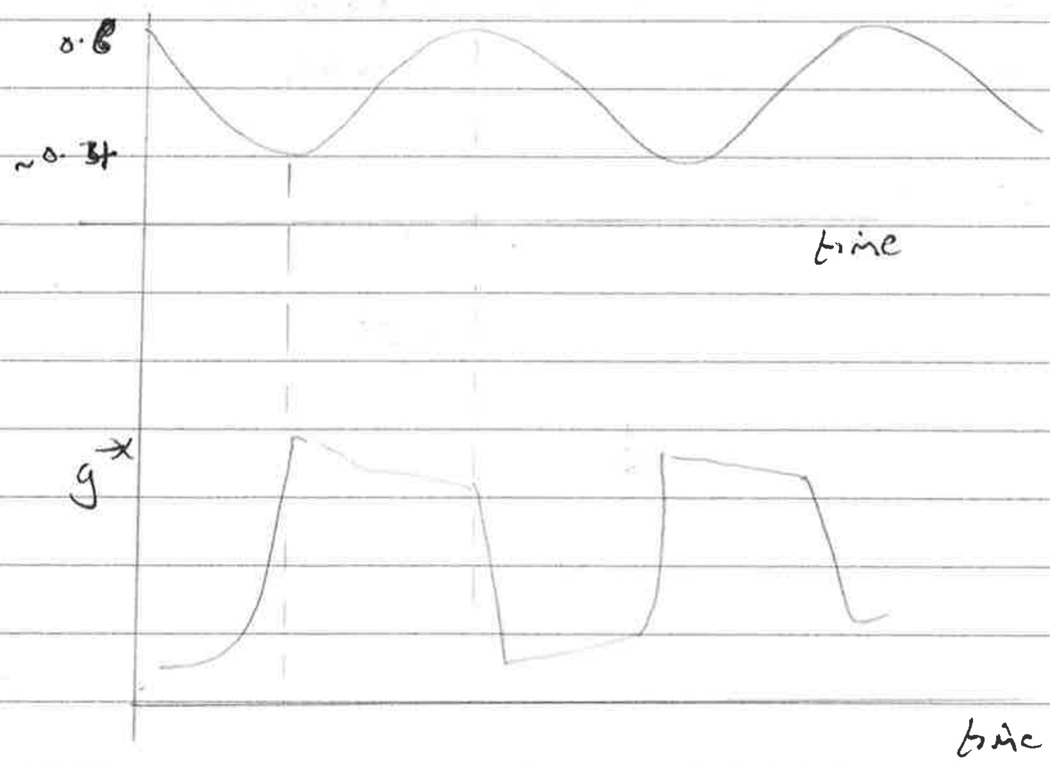
$$\Rightarrow s_{max} = \frac{1}{8} = 0.125$$



Hysteresis = NOT REVERSIBLE, DIFFERENT VALUES OF r AT TRANSITION



Suppose r varies with time



CONCLUSION

The bistable switch is created by the unstable FP between the two stable FPs preventing the flow until a large change in r or a very large fluctuation in g.