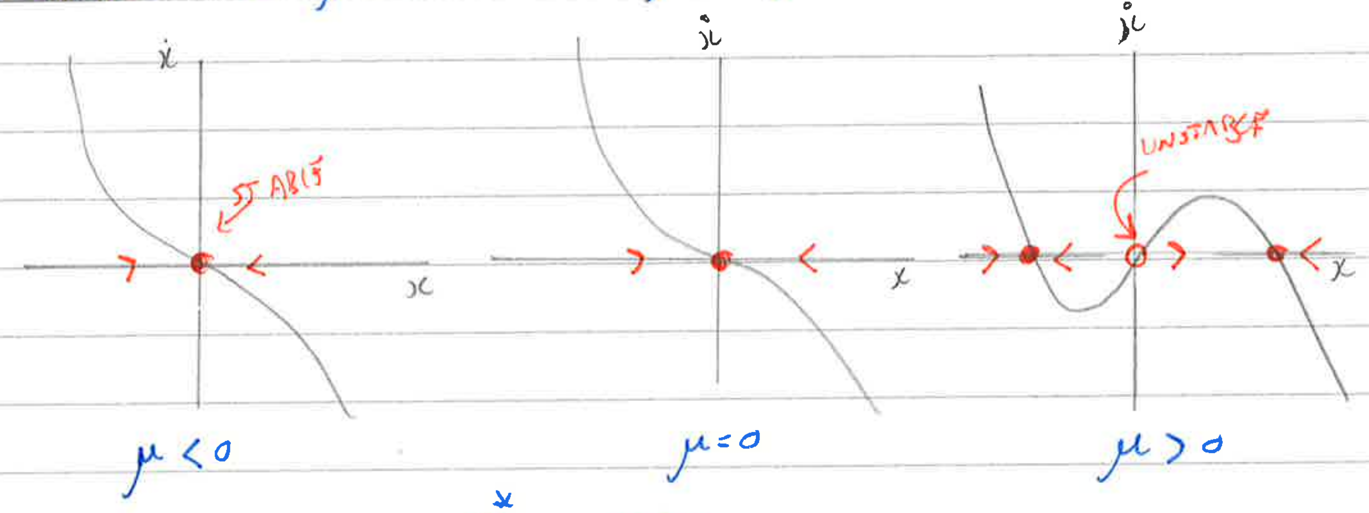
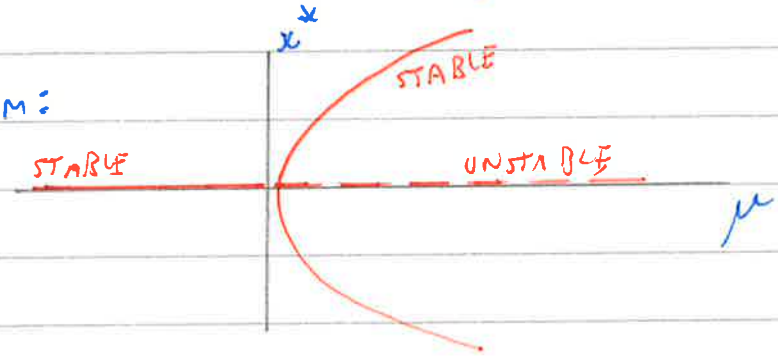
LECTURE 12 : 1D PITCHFORK BIFURCATION (SIDE BLACKBOARD)

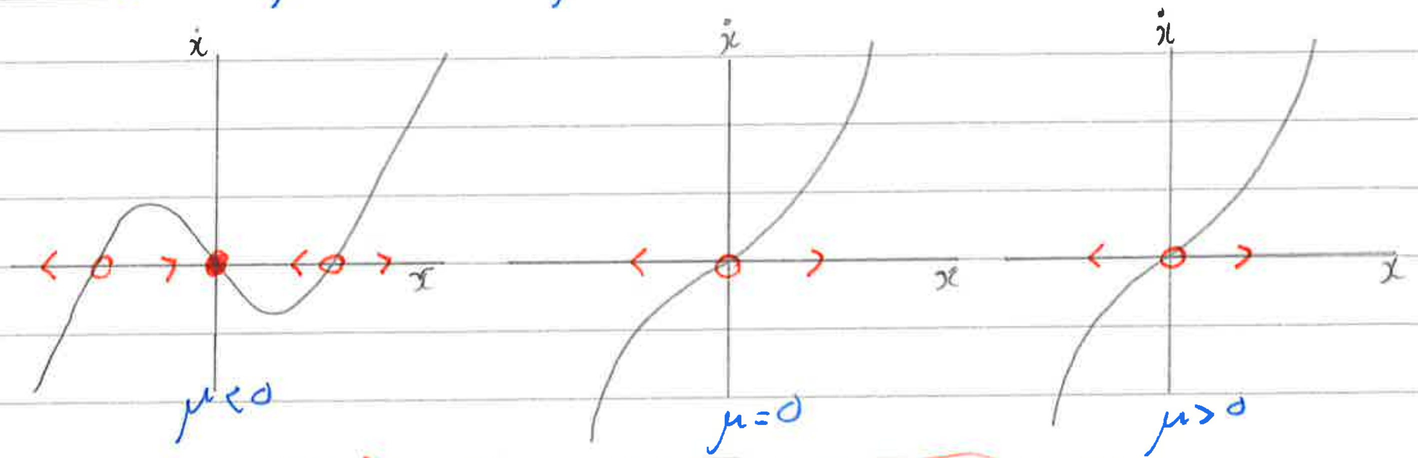
SUPERCRITICAL: $\dot{x} = \mu x - x^3 = x(\mu - x^2)$

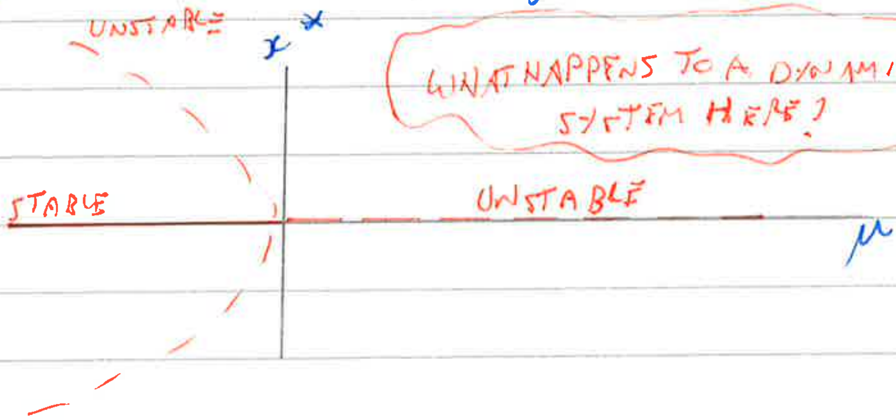


BIFURCATION DIAGRAM:



SUBCRITICAL: $\dot{x} = \mu x + x^3 = x(\mu + x^2)$

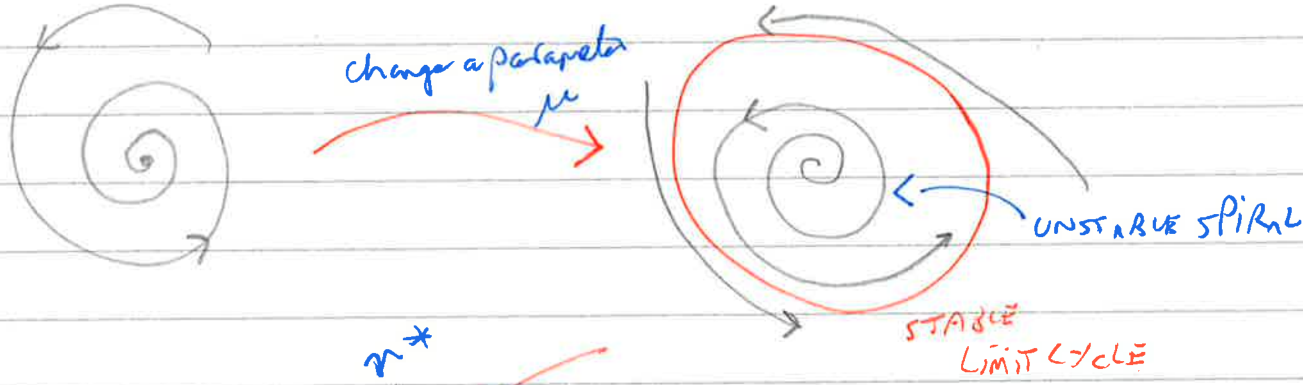


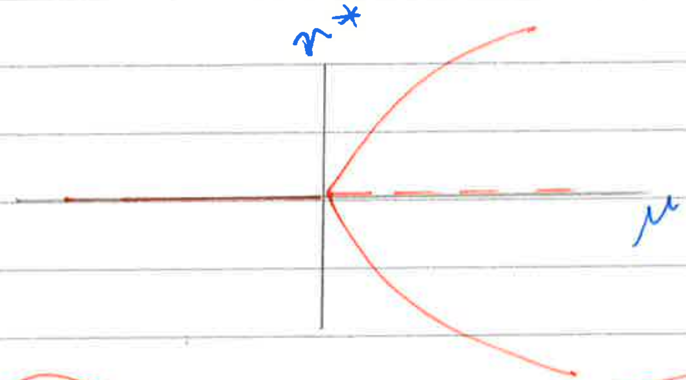


WHAT HAPPENS TO A DYNAMICAL SYSTEM HERE?

Hopf Bifurcations in 2D

Consider a stable spiral in 2D — How can this go unstable as a parameter changes?





2D equivalent of the supercritical pitchfork bifurcation.
i.e. supercritical Hopf bifurcation: aka SOFT
CONTINUOUS
SAFE
} TRANSITION

Recall the dynamical equation in 2D:

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

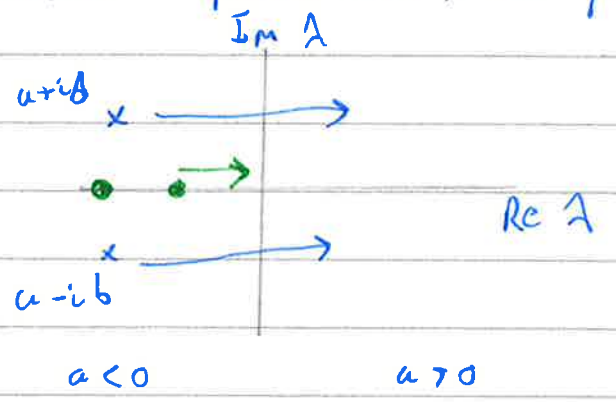
with the Jacobian: $\underline{\underline{J}} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$

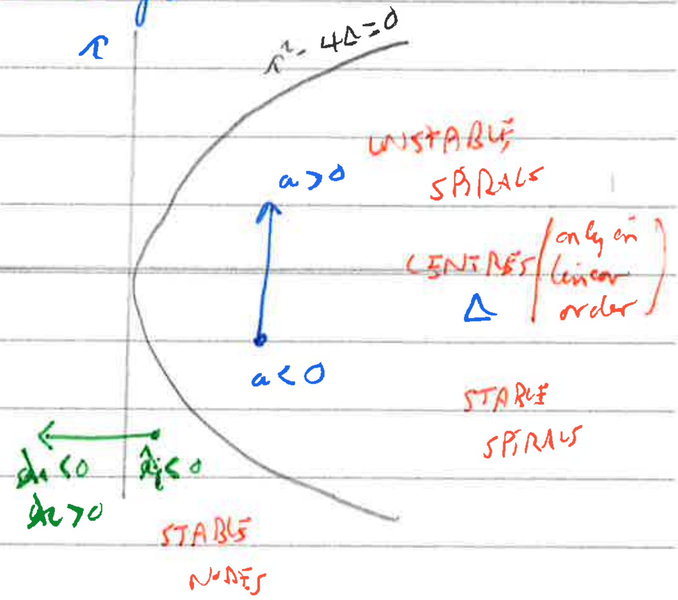
$\underline{J}$ has eigenvalues $\lambda_{1,2} = a \pm i b$ i.e. complex conjugates for a spiral

NB $\tau = \lambda_1 + \lambda_2$ must be real $\Rightarrow$ $\lambda_{1,2}$ are either both real or cc's.

a < 0 stable spiral ($b \neq 0$)

Suppose a goes through 0 as a parameter (μ) changes





then the stable spiral becomes unstable

NB if $b = 0$, both eigenvalues are on Re axis, so if they cross the Im λ axis, one does it first, and it is a saddlenode bifurcation ($\Delta = \lambda_1 \lambda_2 < 0$)

We say a bifurcation has occurred when λ_i cross the Im axis, because there is a qualitative / topological change in the system's dynamics.

So, a stable spiral becomes unstable — is that all that happens? No.

Generalise the pitchfork bifurcation to 2D: plane polar coordinates (r, φ)

$$\dot{r} = \mu r - r^3$$

$$\dot{\varphi} = \omega + b r^2$$

3 Parameters

μ = controls stability
ω = Frequency of small oscillations
b = coupling of frequency to amplitude

NB As $r \to \infty$, $\dot{r} < 0$, so all trajectories head towards the origin

To find the type and stability of the fixed point, we convert to cartesian coordinates

Let $x = r\cos\varphi$, $y = r\sin\varphi$ $\Bigg\{$ $r^2 = x^2 + y^2$, $\tan\varphi = y/x$

$$\Rightarrow \dot{x} = \dot{r}\cos\varphi - r\sin\varphi \cdot \dot{\varphi}$$

$$\dot{y} = \dot{r}\sin\varphi + r\cos\varphi \cdot \dot{\varphi}$$

$$\Rightarrow \dot{x} = (\mu r - r^3)\cos\varphi - r\sin\varphi\,(\omega + b r^2)$$

$$= (\mu - r^2)x - y\,(\omega + b(x^2+y^2))$$

$$= \mu x - (x^2+y^2)x - \omega y - b y (x^2+y^2)$$

$$= \mu x - \omega y - (x + b y)(x^2+y^2)$$

$$\boxed{\therefore \dot{x} = \mu x - \omega y - \underbrace{(x + b y)(x^2+y^2)}_{O(x^3,\, y^3)}}$$

and $\dot{y} = (\mu r - r^3)\sin\phi + x(\omega + b r^2)$

$$= (\mu - (x^2+y^2))\, y + x(\omega + b(x^2+y^2))$$

$$\boxed{\therefore \dot{y} = \omega x + \mu y + (bx - y)(x^2+y^2)}$$

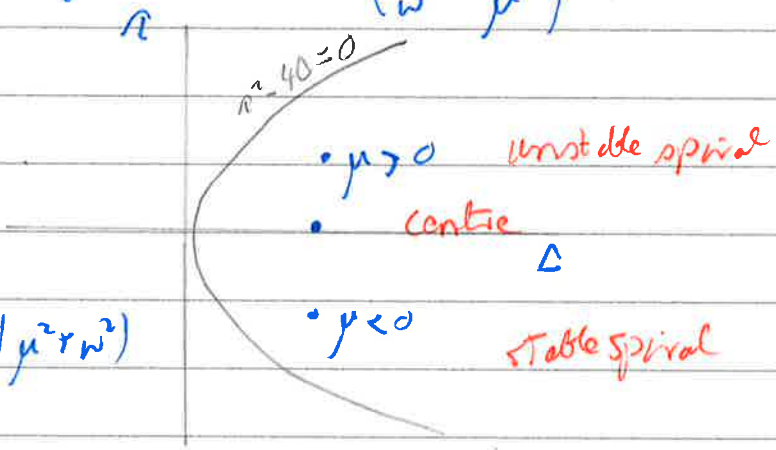
The Jacobian at the origin is: $\underline{J} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$

$\therefore \tau = 2\mu$

$\Delta = \mu^2 + \omega^2$

so $\tau^2 - 4\Delta = (2\mu)^2 - 4(\mu^2+\omega^2)$

$= -4\omega^2 < 0$



$\therefore$ Always a spiral or a centre

Conclusion:

As μ goes from negative to positive, the stable spiral at the origin becomes unstable.

what are the eigenvalues of $\underline{\underline{J}}$?

$$\begin{vmatrix} \mu-\lambda & -\omega \\ \omega & \mu-\lambda \end{vmatrix} = 0$$

$\Rightarrow (\mu-\lambda)^2 + \omega^2 = 0$

$\Rightarrow \mu^2 - 2\mu\lambda + \lambda^2 + \omega^2 = 0$

$\Rightarrow \lambda^2 - 2\mu\lambda + (\mu^2+\omega^2) = 0$

$$\therefore \lambda = \frac{2\mu \pm \sqrt{4\mu^2 - 4(\mu^2+\omega^2)}}{2} = \mu \pm \frac{\sqrt{-4\omega^2}}{2}$$

$$\boxed{\therefore \lambda = \mu \pm i\omega}$$

So, as μ increases through zero, the eigenvalues cross the imaginary axis.

That was the origin, are there other fixed points?

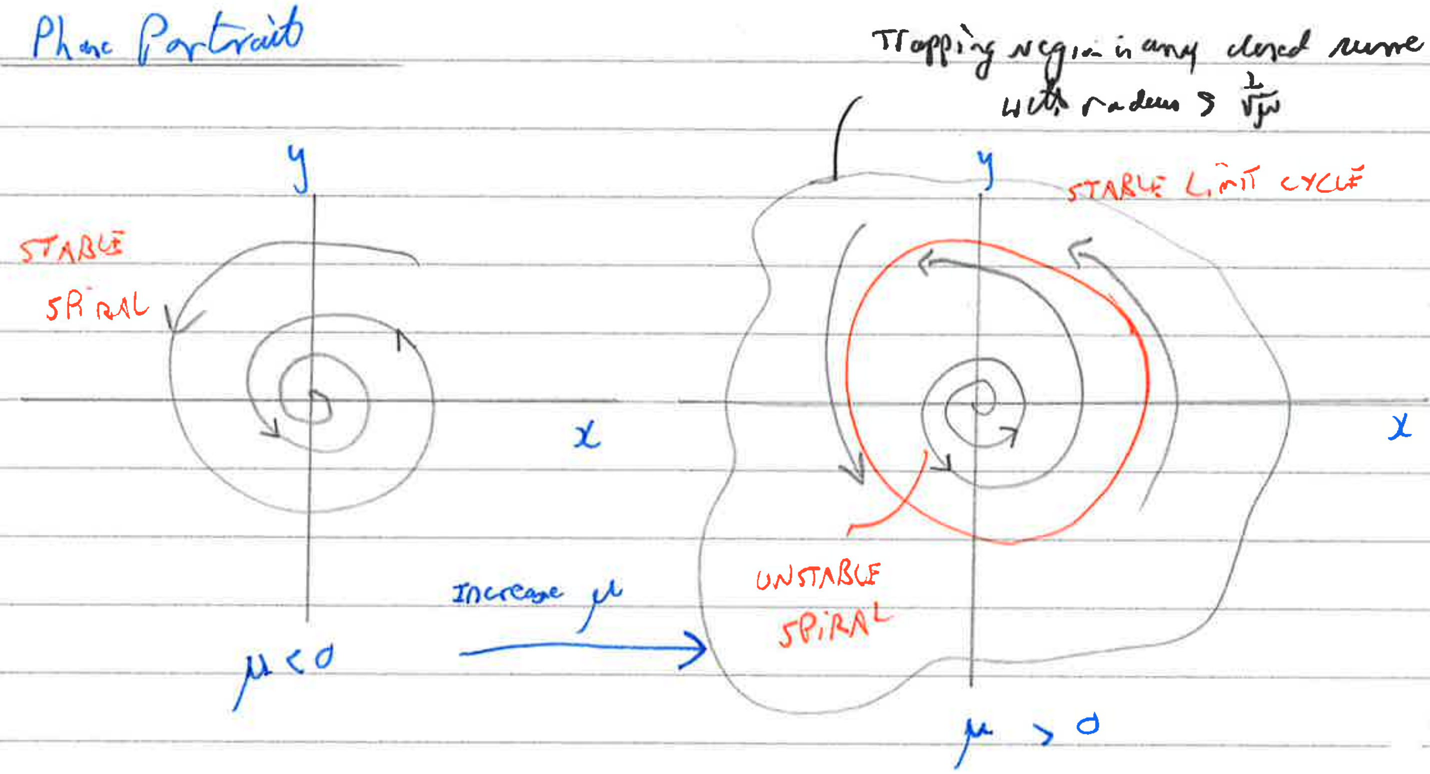
consider the radial equation alone: $\dot{r} = \mu r - r^3 = r(\mu - r^2) = f(r)$

$\therefore r^* = 0, \pm\sqrt{\mu}$ but $r^* < 0$ is meaningless, so only $\boxed{r^* = +\sqrt{\mu} \text{ makes sense.}}$

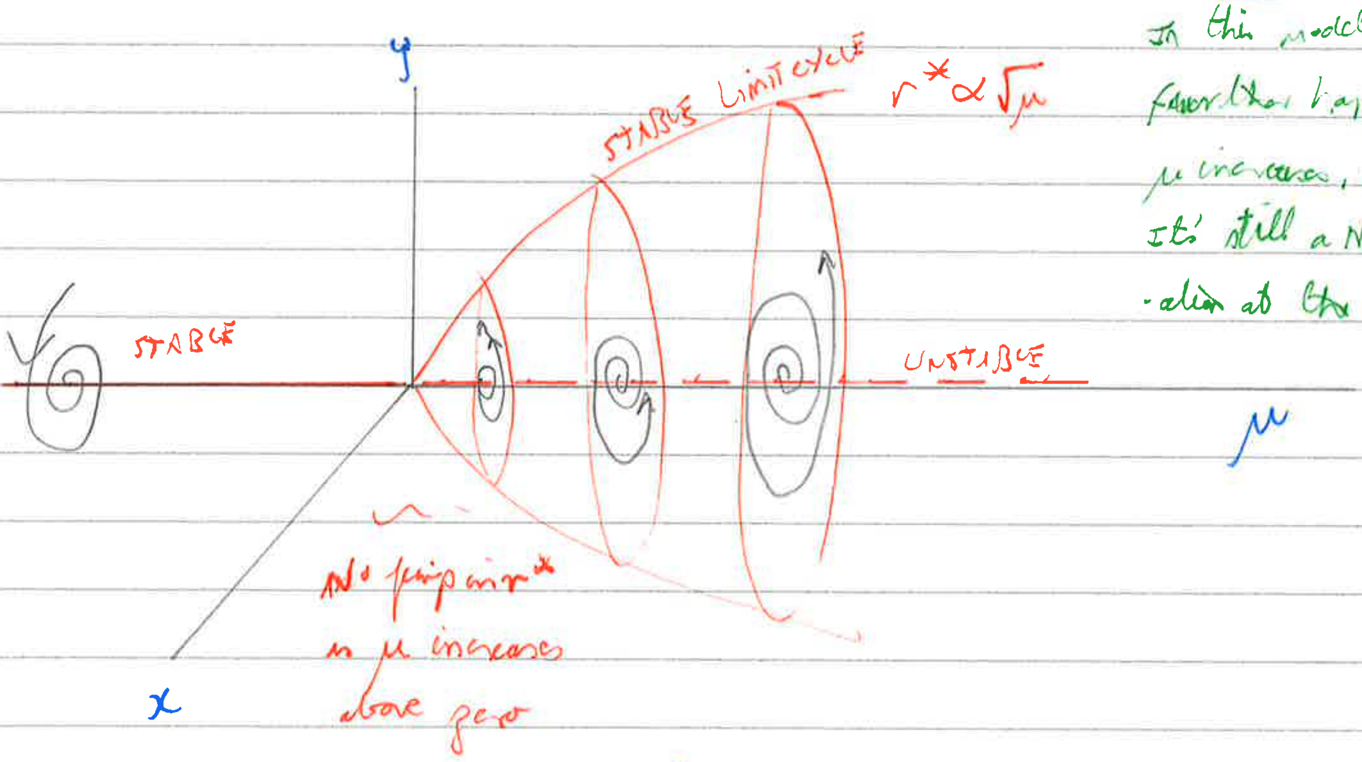
$f'(r) = \mu - 3r^2 \Rightarrow f'(0) = \mu$ i.e. stable if $\mu < 0$, unstable if $\mu > 0$

NB $\dot{\phi} = \omega + br^2 \Rightarrow$ oscillatory motion. $f'(\sqrt{\mu}) = -2\mu$ i.e. unstable if $\mu < 0$, stable if $\mu > 0$.

Phase Portrait



Bifurcation Diagram (cp supercritical Pitchfork $x \leftrightarrow r$)



NB
In this model nothing funny happens as μ increases, but it could; it's still a Hopf bifurcation at the origin.

$\Rightarrow$ SOFT
CONTINUOUS
SAFE } Transition

Summary

As μ increases through zero, the stable spiral at the origin becomes unstable, and a stable limit cycle appears, whose radius grows continuously with μ. Frequency of limit cycle $\sim \omega$ for small amp.

Subcritical Hopf Bifurcation

What happens if the r^3 term is positive? (Draw 5*)

Consider $\dot{r} = \mu r + \underline{r^3} - \underline{r^5}$

$\dot{\varphi} = \omega + b r^2$

r^3 now destabilise the system, and $-r^5$ stabilises it at larger r!

we still have $x \to -x$ symmetry.

Rewrite in Cartesian coordinates:

$$\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right\} \begin{array}{l} r^2 = x^2+y^2 \\ \tan\varphi = \frac{y}{x} \end{array}$$

$$\Rightarrow \dot{x} = \dot{r}\cos\varphi - r\sin\varphi \cdot \dot{\varphi}$$

$$= (\mu \overset{\equiv x}{r} + r^3 - r^5)\cos\varphi - \underbrace{r\sin\varphi}_{\equiv y}\,(\omega + b r^2)$$

$$= x(\mu + r^2 - r^4) - y(\omega + b r^2)$$

$$= \mu x - \omega y + (x - by) r^2 - x r^4$$

$$\boxed{\therefore\ \dot{x} = \mu x - \omega y + (x-by)(x^2+y^2) - x(x^2+y^2)^2}$$

Similarly with y:

$$\dot{y} = \dot{r}\sin\varphi + r\cos\varphi\,\dot{\varphi}$$

$$= (\mu r + r^3 - r^5)\sin\varphi + r\cos\varphi(\omega + b r^2)$$

$$\Rightarrow \dot{y} = (\mu + r^2 - r^4)\, y + x\,(\omega + b r^2)$$

$$= \mu y + \omega x + y\left[(x^2+y^2) - (x^2+y^2)^2\right] + bx(x^2+y^2)$$

$$\boxed{\therefore \dot{y} = \mu y + \omega x + (bx+y)(x^2+y^2) - y(x^2+y^2)^2}$$

The Jacobian is again: $\underline{\underline{J}} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$

Same as supercritical case!

NB Linear Stability Analysis cannot distinguish super- from subcritical Hopf.

~~[illegible]~~

so, $T = 2\mu$

$\Delta = \mu^2 + \omega^2$

and again we have a stable spiral at the origin when $\mu < 0$.

What happens when $\mu > 0$? Where are the fixed points?

$$\dot{r} = \mu r + r^3 - r^5 = r(\mu + r^2 - r^4) = -r(r^4 - r^2 - \mu)$$

Let $x = r^2$, so we have: $x^2 - x - \mu = 0$

$$\Rightarrow x = \frac{1 \pm \sqrt{1+4\mu}}{2}$$

so the roots of $\dot{r}$ are: $\boxed{r^* = 0,\ (r_\pm^*)^2 = \dfrac{1 \pm \sqrt{1+4\mu}}{2}}$

$$\underline{\text{so } \dot{r} = -r(r^2 - r_-^2)(r^2 - r_+^2)}$$

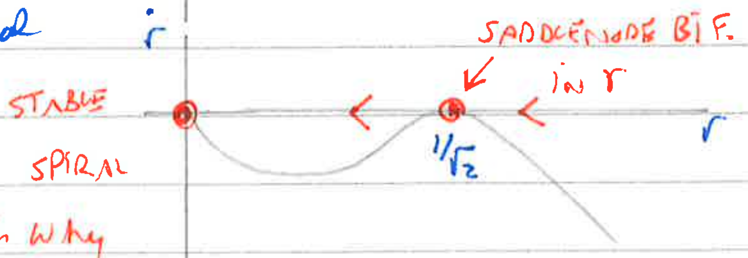
note the sign of r^5 term is negative as it should be.

$$r_\pm^2 = \frac{1 \pm \sqrt{1+4\mu}}{2}$$

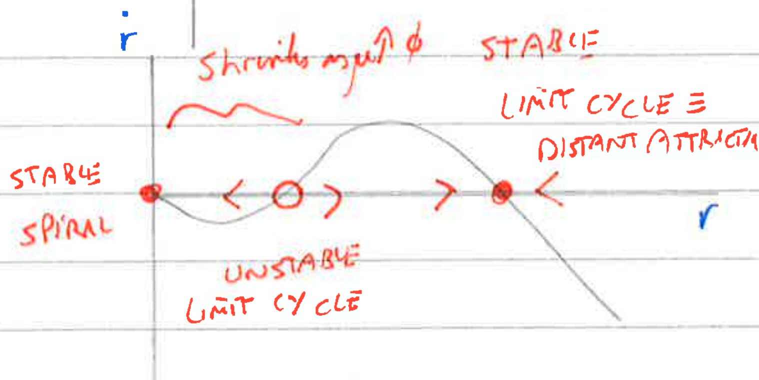
What does the vector field $\dot{r}$ look like as μ changes?

$\mu < -1/4$, $r_\pm$ are complex, so only $r=0$ is real
$F'(0) = \mu - 3r^2 - 4r^4 = \mu < 0$ $\therefore$ stable

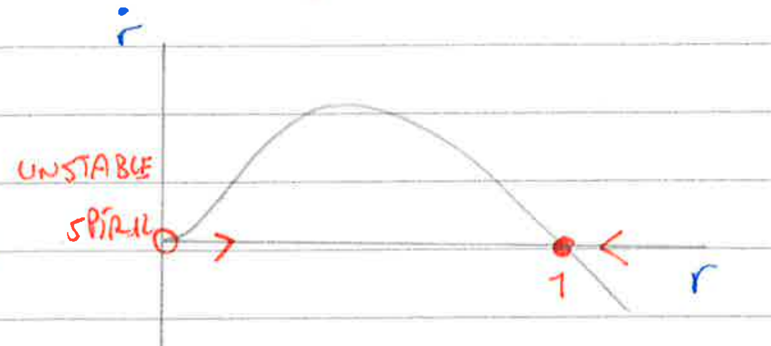
$\mu = -1/4$, $r_\pm = \pm 1/\sqrt{2}$, so only $0, 1/\sqrt{2}$ real and physically relevant.
Notice that $r_+ = 1/\sqrt{2}$ does not depend on μ, so it doesn't grow from zero as μ increases: this is why the transition is called DISCONTINUOUS



$-1/4 < \mu < 0$, $r_\pm$ are real and distinct, and a stable and unstable limit cycle appear around the stable spiral by a saddlenode bifurcation.

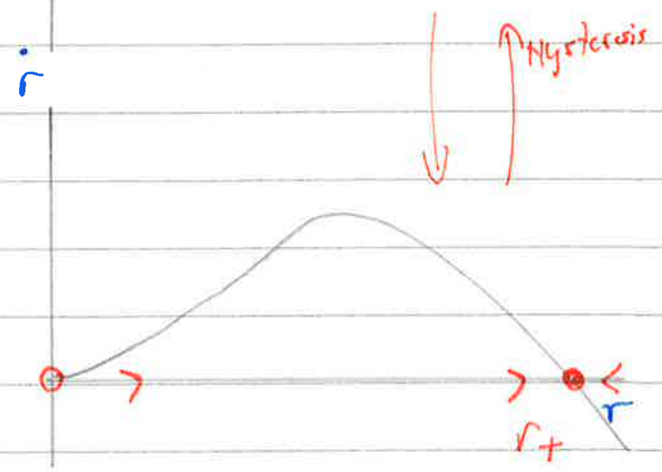


$\mu = 0$, $r_\pm = 0, 1$
The origin is now an unstable spiral, surrounded by a limit cycle at large r.
i.e. $r^* \sim O(1)$ not $O(\sqrt{\mu})$.



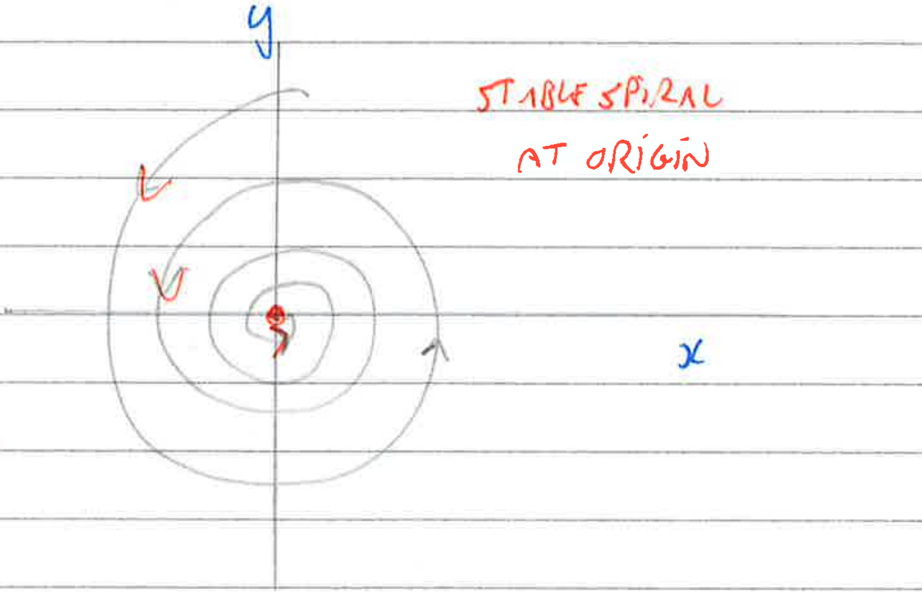
$\mu > 0$, $r_+ = \frac{1+\sqrt{1+4\mu}}{2}$, r_- is complex

Same as $\mu = 0$ but the stable limit cycle moves out to larger radius.

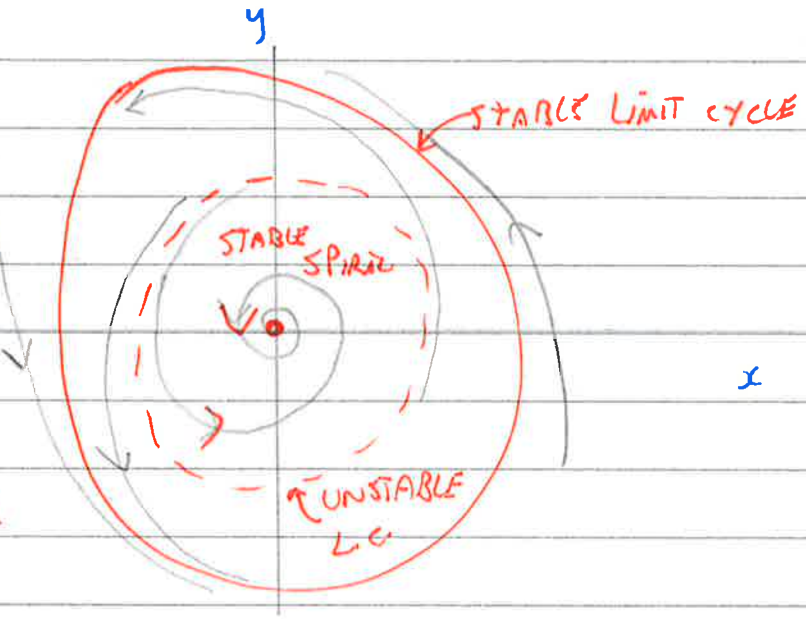


What do the trajectories look like?

$\mu \le -1/4$



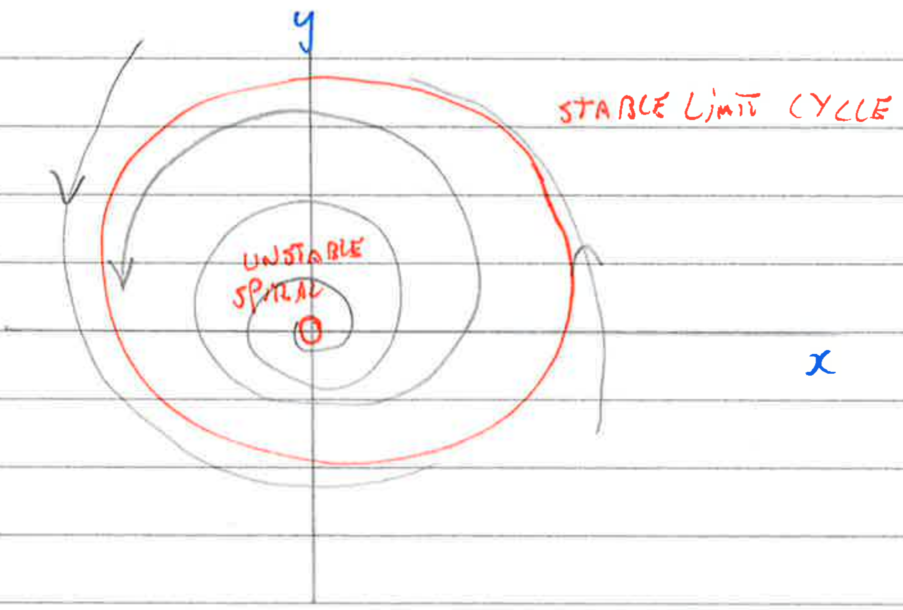
$-1/4 < \mu < 0$



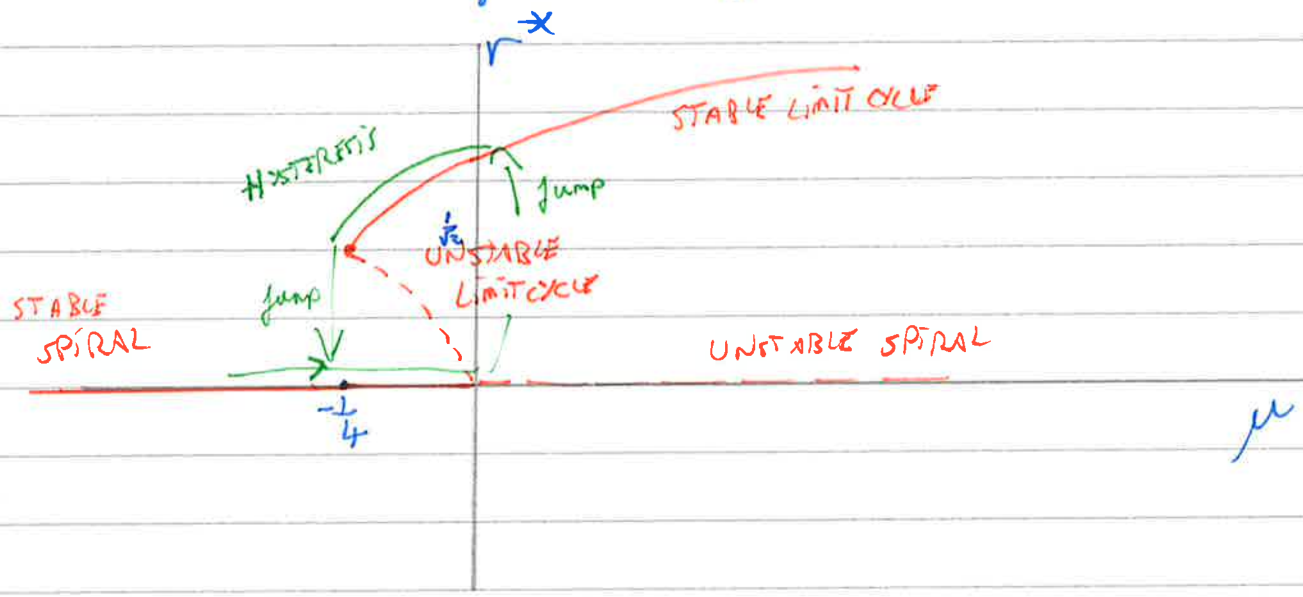
The unstable limit cycle shrinks onto the stable spiral at origin and makes it unstable.

NB Radii are just roots of the cubic $\dot r$ equation.

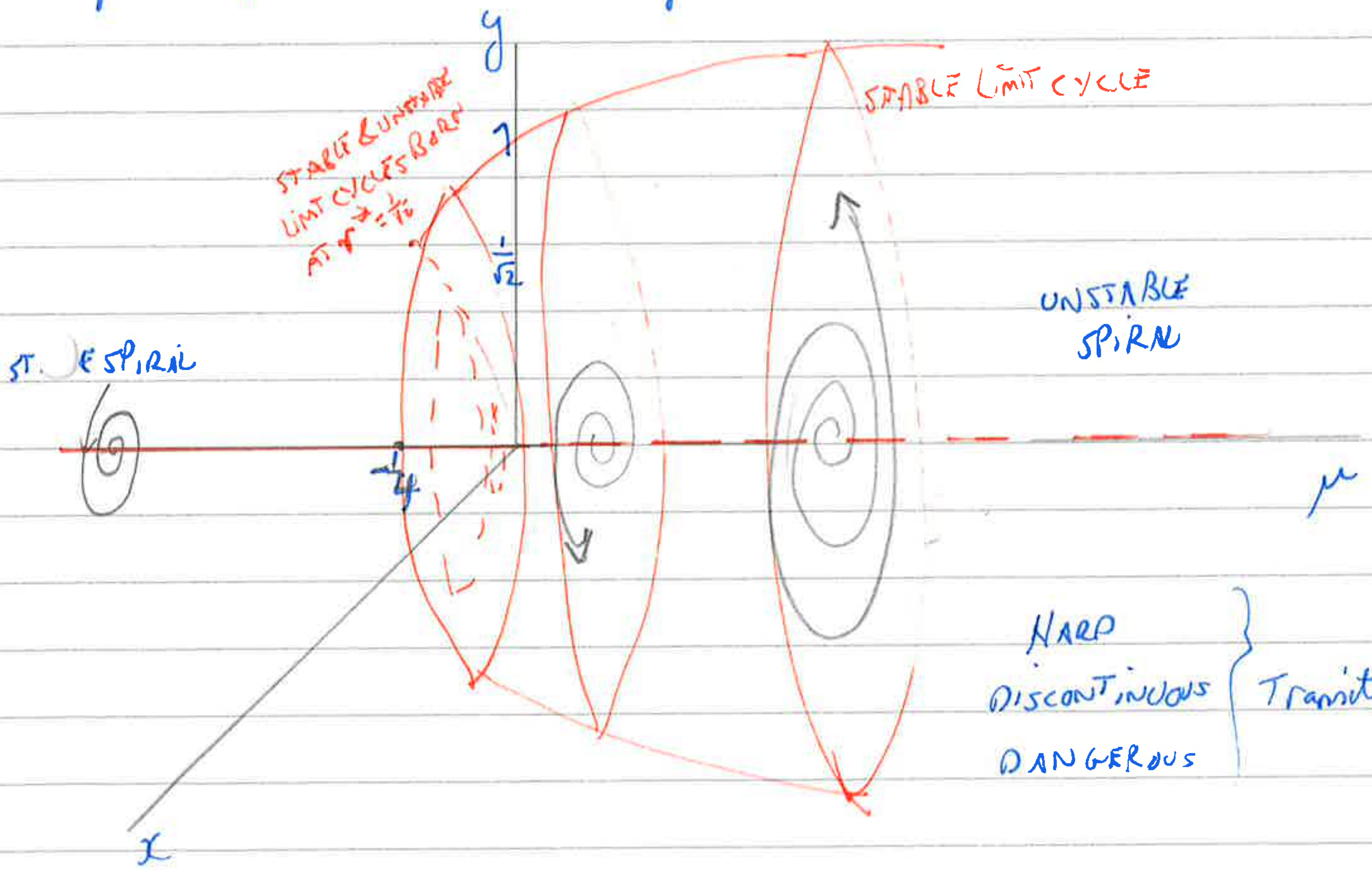
$\mu > 0$



What does the bifurcation diagram look like?



Phase portrait for Subcritical Hopf bifurcation



HARD
DISCONTINUOUS } Transition
DANGEROUS

Summary. Because the stable limit cycle for $\mu \geq -1/4$ starts with radius $\sim \frac{1}{\sqrt{2}}$ (and $\varepsilon \approx c$) the system moves to a large distance away from origin; this is why it is dangerous on $\mu \geq 0$. Also hysteresis prevents the large amplitude oscillations dying away until μ is reduced a long way.