Introduction to Quantum Science and Technology

Final exam
Fall term 2024

Assignment date: **thursday January 30, 2025, 9h15 - 12h15**
QUANT 400 – Exam – room BC 01

- There are 5 problems with equal weight. You have 3 hours.
- **For each problem, write your solutions in the indicated space.** Scrap paper will not be corrected.
- Turn off wifi of in your electronic devices. Use of a calculator is not allowed.
- **Don't forget to clearly write your name below as well as on other problem sheets.**
- Good luck!

Name: ____________________________

Section: ______________

Sciper No.: ______________

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Total	/100

Problem 1. *Nicolas Macris* (total: 20 marks)

Student Name: ____________________________

Section: ______________

Sciper No.: ______________

We recall the standard teleportation protocol. Two parties A and B share a perfect Bell pair $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}$. Alice also holds another qubit in state $|\psi\rangle_{A'} = \alpha|0\rangle + \beta|1\rangle)$. Firstly, A applies a Bell basis measurement to its two qubits. Secondly, A sends two classical bits to B informing which of four measurement outcomes occurred. Thirdly, B applies an appropriate unitary operation (depending on the two classical bits received) and obtains the wanted teleported state.

We also recall the Bell basis states $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

1. **Teleportation of mixed states.** Suppose A and B want to teleport a mixed state $\sigma_{A'} = \sum_i q_i |\psi_i\rangle_{A'}\langle\psi_i|_{A'}$ where $|\psi_i\rangle_{A'} = \alpha_i|0\rangle + \beta_i|1\rangle$.
   (a) What are the possible measurement outcomes in A-lab ?
   (b) For each measurement outcome in A-lab, what are the possible states for the qubit in B-lab ? (You can give the result without showing calculations)
   (c) What is the final teleported state B-lab at the very end of the protocol ? (No calculation required, but justify your reasoning)

2. **Teleportation when the entangled link is broken.** Now we suppose that the entangled link $|\Phi^+\rangle_{AB}$ is broken (by too much noise say) and is replaced by the *density matrix* $\frac{I_A}{2} \otimes \frac{I_B}{2}$ where I_A and I_B are 2×2 identity matrices.
   (a) If $\sigma_{A'}$ is the state to be teleported, what is the state at the end of the protocol that B holds ?
   (b) Suppose now that the entangled link is a mixture of a perfect and completely noisy link (the so-called Werner state)

$$\rho_{AB} = (1-\epsilon)\,|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB} + \epsilon\,\frac{I_A}{2} \otimes \frac{I_B}{2}, \qquad 0 \le \epsilon \le 1.$$

   Again, if $\sigma_{A'}$ is the state to be teleported, what is the state at the very end of the protocol in the B-lab ?

3. **A one-dimensional communication network.** Consider N nodes $X_1, X_2, X_3, X_4, \cdots$ arranged on a line. Each adjacent pair of nodes (X_n, X_{n+1}) shares a pair of qubits in a mixed Werner state described by the density matrix

$$\rho_{n,n+1} = (1-\epsilon)\,|\Phi^+\rangle\langle\Phi^+| + \epsilon\,\frac{I}{2} \otimes \frac{I}{2}, \quad 0 \le \epsilon \le 1$$

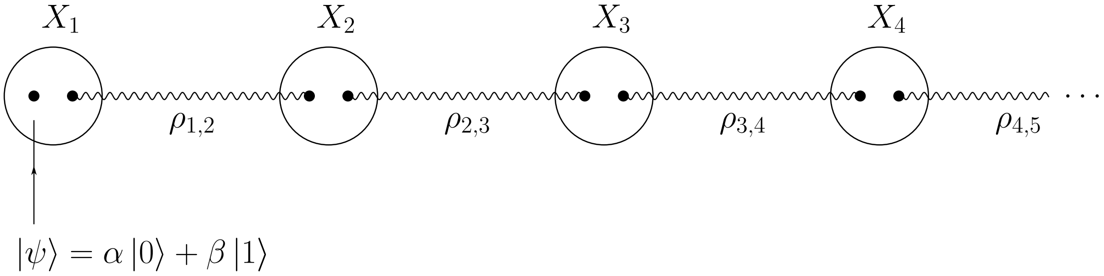
See the figure where each bond between (X_n, X_{n+1}) represents the Werner state $\rho_{n,n+1}$. The node X_1 holds an extra qubit in a pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

The goal is to teleport this state through the noisy network. See figure.

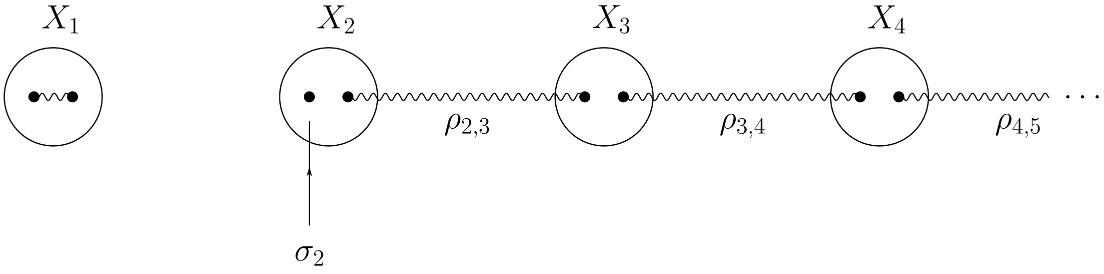
(a) Compute the first few teleported states accross the network and deduce the teleported state density matrix σ_n at node n in terms of $|\psi\rangle\langle\psi|$ and ϵ.

(b) Compute the fidelity $F_n = \mathrm{Tr}(\sigma_n |\psi\rangle\langle\psi|)$ and the limit $\lim_{n\to+\infty} F_n$.

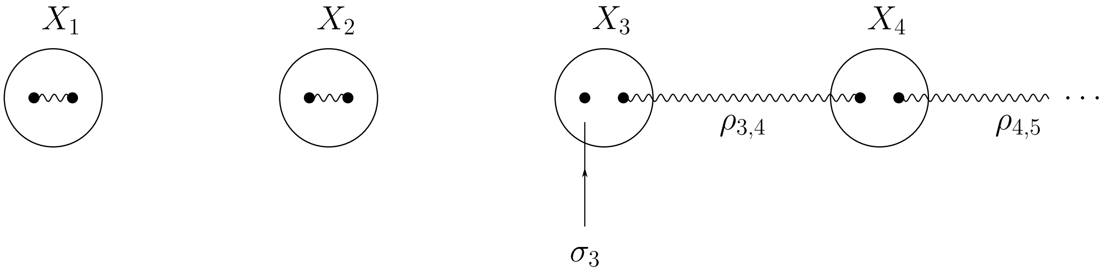
Before the first teleportation step:



After the first teleportation step:



After the second teleportation step:



Solution to Problem 1:

1.a) *(1 mark).* The possible outcomes in A-lab are just the four Bell basis states $|\Phi^+\rangle_{AA'}$, $|\Phi^-\rangle_{AA'}$, $|\Psi^+\rangle_{AA'}$, $|\Psi^-\rangle_{AA'}$.

1.b) *(2 marks).* With probability q_i the state to be teleported is $|\psi_i\rangle$. Thus we have

- measurement outcome $|\Phi^+\rangle_{AA'}$ gives $\alpha_i|0\rangle_B + \beta_i|1\rangle_B$.
- measurement outcome $|\Phi^-\rangle_{AA'}$ gives $\alpha_i|0\rangle_B - \beta_i|1\rangle_B = Z|\psi_i\rangle$.
- measurement outcome $|\Psi^+\rangle_{AA'}$ gives $\alpha_i|1\rangle_B + \beta_i|0\rangle_B = X|\psi_i\rangle$.
- measurement outcome $|\Psi^-\rangle_{AA'}$ gives $\alpha_i|1\rangle_B - \beta_i|0\rangle_B = XZ|\psi_i\rangle\rangle$.

Now, by linearity of this 'map' the corresponding states in the B-lab are the density matrices: $\sigma_B = \sum_i q_i|\psi_i\rangle_B\langle\psi_i|_B$, $Z\sigma_B Z$, $X\sigma_B X$, $XZ\sigma_B ZX$.

1.c) *(2 marks).* B applies the appropriate unitary operation for each of the four case: I_B, Z_B, X_B, $Z_B X_B$, to always recover σ_B. Note that mathematically the unitary is applied as a conjugation of the density matrices.

2.a) *(4 marks).* The initial state is $\sigma_{A'} \otimes \frac{A}{2} \otimes \frac{B}{2}$. After the measurement in A lab the state is one of

$$|\text{Bell}\rangle\langle\text{Bell}|\sigma_{A'} \otimes \frac{I_A}{2}|\text{Bell}\rangle\langle\text{Bell}| \otimes \frac{I_B}{2}$$

where $|\text{Bell}\rangle$ is one of the four Bell states. For each measurement outcome, in the B-lab the state is $\frac{I_B}{2}$. When B receives the classical message the appropriate unitary is applied, and this identity matrix stay the identity matrix, so the 'teleported' state is $\frac{I_B}{2}$.

2.b) *(4 marks).* With probability $1-\epsilon$ the density matrix $\sigma_{A'}$ is perfectly teleported and σ_B is obtained in B-lab. With probability ϵ from the previous question the teleported state is obtained as $\frac{I_B}{2}$. Thus by linearity of the 'map' we have the teleported state

$$(1-\epsilon)\sigma_B + \epsilon\frac{I_B}{2}$$

3.a) *(5 marks).* From previous questions we see that, when $|\psi\rangle\langle\psi|$ is teleported from first to second node, the second node gets

$$\sigma_2 = (1-\epsilon)|\psi\rangle\langle\psi| + \epsilon\frac{I}{2}$$

Then, when this state is teleported from second to third node the third node gets

$$\begin{aligned}\sigma_3 &= (1-\epsilon)\sigma_2 + \epsilon\frac{I}{2} \\ &= (1-\epsilon)^2|\psi\rangle\langle\psi| + (2\epsilon-\epsilon^2)\frac{I}{2} \\ &= (1-\epsilon)^2|\psi\rangle\langle\psi| + (1-(1-\epsilon)^2)\frac{I}{2} \qquad (1)\end{aligned}$$

Iterating we find

$$\sigma_n = (1-\epsilon)^{n-1}|\psi\rangle\langle\psi| + (1-(1-\epsilon)^{n-1})\frac{I}{2}$$

Therefore we have

$$\gamma_n = 1-(1-\epsilon)^{n-1}$$

Sanity check: $\gamma_1 = 0$, $\gamma_2 = \epsilon$ as it should.

3.b) *(2 marks).* The fidelity is

$$\begin{aligned} F_n &= (1-\epsilon)^{n-1} + \frac{1}{2}(1-(1-\epsilon)^{n-1}) \\ &= \frac{1}{2} + \frac{1}{2}(1-\epsilon)^{n-1} \end{aligned} \tag{2}$$

At $n = 1$ the fidelity is maximal equal to 1 and as $n \to +\infty$ it tends to $1/2$ (worst fidelity) exponentially fast.

Problem 2. *Giuseppe Carleo* (total: 20 marks)

Student Name: ____________________________

Section: ______________

Sciper No.: ______________

We consider the two-qubit Hamiltonian:

$$\widehat{H} \;=\; \sigma_x \otimes \sigma_x \;+\; \sigma_y \otimes \sigma_y \;+\; \sigma_z \otimes \sigma_z,$$

where $\sigma_x, \sigma_y, \sigma_z$ are the usual single-qubit Pauli matrices defined as:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Analytical Properties

1. Write $\widehat{H}$ explicitly in the computational basis $\{|00\rangle,\ |01\rangle,\ |10\rangle,\ |11\rangle\}$. Verify that $\widehat{H}$ is Hermitian.

2. Find the *eigenvalues* of $\widehat{H}$ and show that its *eigenstates* are the four Bell states (up to overall phases). As a reminder, the 4 Bell states are:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big), \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big),$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big).$$

3. Suppose the system is prepared in the state $|01\rangle$ at $t = 0$.
   (a) Expand $|01\rangle$ in the *Bell basis* (i.e. the basis of the 4 Bell states).
   (b) Use this expansion to determine $|\psi(t)\rangle = e^{-\mathrm{i}\widehat{H}t}|01\rangle$ as a function of t and in the computational basis (Tip: you may ignore any overall global phase)

(b) Time-Evolution Operator and Trotterization

We want to implement

$$U(t) \;=\; \exp\big(-\mathrm{i}\,\widehat{H}\,t\big)$$

on a quantum computer, starting from $|01\rangle$.

1. Rember that
   $$\widehat{H} \;=\; \big(\sigma_x \otimes \sigma_x \;+\; \sigma_y \otimes \sigma_y \;+\; \sigma_z \otimes \sigma_z\big).$$
   Propose a *grouping* of these terms that you will use for Trotterization (no need to show error bounds).

2. For the exponentials
   $$\exp\big(-\mathrm{i}\,\sigma_x \otimes \sigma_x\,\Delta t\big), \quad \exp\big(-\mathrm{i}\,\sigma_y \otimes \sigma_y\,\Delta t\big), \quad \exp\big(-\mathrm{i}\,\sigma_z \otimes \sigma_z\,\Delta t\big),$$
   describe how each can be realized with *standard two-qubit gates* (e.g. single-qubit rotations, CNOTs).

3. Assume you apply a single Trotter step to the initial state $|01\rangle$. Compute the *probability* of measuring the *first qubit* in $|0\rangle$ *as a function of* the time-slice discretization Δt.

Solution to Problem 2:

Solution to Part (a)

(a.1) Matrix Representation

We have the Hamiltonian

$$\widehat{H} \;=\; \Big(\sigma_x \otimes \sigma_x \;+\; \sigma_y \otimes \sigma_y \;+\; \sigma_z \otimes \sigma_z\Big),$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, each term expands to a 4×4 matrix:

$$\sigma_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Summing the three terms gives:

$$\sigma_x \otimes \sigma_x \;+\; \sigma_y \otimes \sigma_y \;+\; \sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so

$$\widehat{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Clearly, this matrix is *real-symmetric*, hence Hermitian.

(a.2) Eigenvalues and Bell States

Let us verify that the Bell states are eigenstates of $\hat{H}$ by direct calculation:

For $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$:

$$\begin{aligned}
\hat{H}|\Phi^+\rangle &= \frac{1}{\sqrt{2}}\hat{H}(|00\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}[(1)|00\rangle + (1)|11\rangle] \\
&= (+1)|\Phi^+\rangle
\end{aligned}$$

For $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$:

$$\begin{aligned}
\hat{H}|\Phi^-\rangle &= \frac{1}{\sqrt{2}}\hat{H}(|00\rangle - |11\rangle) \\
&= \frac{1}{\sqrt{2}}[(1)|00\rangle - (1)|11\rangle] \\
&= (+1)|\Phi^-\rangle
\end{aligned}$$

For $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$:

$$\begin{aligned}
\hat{H}|\Psi^+\rangle &= \frac{1}{\sqrt{2}}\hat{H}(|01\rangle + |10\rangle) \\
&= \frac{1}{\sqrt{2}}[(-1|01\rangle + 2|10\rangle) + (2|01\rangle - 1|10\rangle)] \\
&= \frac{1}{\sqrt{2}}[(|01\rangle + |10\rangle)] \\
&= (+1)|\Psi^+\rangle
\end{aligned}$$

For $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$:

$$\begin{aligned}
\hat{H}|\Psi^-\rangle &= \frac{1}{\sqrt{2}}\hat{H}(|01\rangle - |10\rangle) \\
&= \frac{1}{\sqrt{2}}[(-1|01\rangle + 2|10\rangle) - (2|01\rangle - 1|10\rangle)] \\
&= \frac{1}{\sqrt{2}}[(-3|01\rangle + 3|10\rangle)] \\
&= (-3)|\Psi^-\rangle
\end{aligned}$$

Therefore, the eigenvalues are:

- $\lambda_1 = +1$ (triply degenerate) with eigenstates $|\Phi^+\rangle$, $|\Phi^-\rangle$, and $|\Psi^+\rangle$
- $\lambda_2 = -3$ with eigenstate $|\Psi^-\rangle$

(a.3) Time Evolution of $|01\rangle$

(a) The state $|01\rangle$ can be expanded in the Bell basis as:

$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle)$$

(b) Given that $|\Psi^+\rangle$ has eigenvalue $+1$ and $|\Psi^-\rangle$ has eigenvalue -3, the time evolution is:

$$\begin{aligned}|\psi(t)\rangle &= e^{-i\hat{H}t}|01\rangle \\ &= \frac{1}{\sqrt{2}}(e^{-i(+1)t}|\Psi^+\rangle + e^{-i(-3)t}|\Psi^-\rangle) \\ &= \frac{1}{\sqrt{2}}(e^{-it}|\Psi^+\rangle + e^{+3it}|\Psi^-\rangle)\end{aligned}$$

Using the Bell-state expansions:

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

we substitute back into $|\psi(t)\rangle$:

$$\begin{aligned}|\psi(t)\rangle &= \frac{1}{\sqrt{2}}\left[e^{-it}\left(\frac{1}{\sqrt{2}}[|01\rangle + |10\rangle]\right) + e^{+3it}\left(\frac{1}{\sqrt{2}}[|01\rangle - |10\rangle]\right)\right] \\ &= \frac{1}{2}\left[e^{-it}(|01\rangle + |10\rangle) + e^{+3it}(|01\rangle - |10\rangle)\right]\end{aligned}$$

Grouping terms explicitly:

$$|\psi(t)\rangle = \frac{1}{2}\left[(e^{-it} + e^{+3it})|01\rangle + (e^{-it} - e^{+3it})|10\rangle\right]$$

Often, we drop the overall phase e^{-it} (because global phases do not affect measurement outcomes). If you factor out e^{-it}, you get

$$|\psi(t)\rangle \sim \frac{e^{-it}}{2}\left[(1 + e^{+4it})|01\rangle + (1 - e^{+4it})|10\rangle\right]$$

where "$\sim$" means "equal up to a global phase." In either form, we now have $|\psi(t)\rangle$ explicitly in the standard $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ basis.

Solution to Part (b)

We wish to implement

$$U(t) \;=\; \exp\bigl(-\,i\,\widehat{H}\,t\bigr) \quad \text{where} \quad \widehat{H} \;=\; \sigma_x \otimes \sigma_x \;+\; \sigma_y \otimes \sigma_y \;+\; \sigma_z \otimes \sigma_z.$$

Because the three terms $\sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_y$, and $\sigma_z \otimes \sigma_z$ *do not commute* with each other, we must use a *Trotter* (or Suzuki–Trotter) decomposition to approximate the full evolution by products of exponentials of individual terms over small time slices.

(b.1) Grouping Non-Commuting Terms

First-Order Trotter. Divide the total time t into N slices of length $\Delta t = t/N$. Then we approximate:

$$\exp\bigl(-\,i\,\widehat{H}\,\Delta t\bigr) \;\approx\; \exp\bigl(-i\,\sigma_x \otimes \sigma_x\,\Delta t\bigr)\;\exp\bigl(-i\,\sigma_y \otimes \sigma_y\,\Delta t\bigr)\;\exp\bigl(-i\,\sigma_z \otimes \sigma_z\,\Delta t\bigr),$$

where the overall minus sign in $\widehat{H}$ has been absorbed as $+i$ in each exponential. Repeating this step N times yields $\bigl(U_{\Delta t}\bigr)^N \approx \exp(-\,i\,\widehat{H}\,t)$.

Second-Order Trotter–Suzuki. To reduce the Trotter error, one can interleave "half-steps" in a symmetric way:

$$U^{(2)}_{\Delta t} \;\approx\; \exp\bigl(-i\,\sigma_x{\otimes}\sigma_x\,\tfrac{\Delta t}{2}\bigr)\;\exp\bigl(-i\,\sigma_y{\otimes}\sigma_y\,\tfrac{\Delta t}{2}\bigr)\;\exp\bigl(-i\,\sigma_z{\otimes}\sigma_z\,\Delta t\bigr)\;\exp\bigl(-i\,\sigma_y{\otimes}\sigma_y\,\tfrac{\Delta t}{2}\bigr)\;\exp\bigl(-i\,\sigma_x{\otimes}\sigma_x\,\tfrac{\Delta t}{2}\bigr),$$

then repeat this block N times with $\Delta t = t/N$. Such symmetry cancels certain lower-order errors.

(b.2) Implementing $\exp\bigl(-i\,\sigma_\alpha \otimes \sigma_\alpha\,\Delta t\bigr)$

Each exponential $\exp\bigl(-i\,\sigma_\alpha \otimes \sigma_\alpha\,\Delta t\bigr)$ can be realized by:

1. **Basis Change.** Convert $\sigma_\alpha \otimes \sigma_\alpha$ to $\sigma_z \otimes \sigma_z$ via single-qubit rotations:

   $$H\,\sigma_x\,H = \sigma_z, \quad R_x\bigl(\tfrac{\pi}{2}\bigr)\,\sigma_y\,R_x\bigl(-\tfrac{\pi}{2}\bigr) = \sigma_z,$$

   etc. For instance, for $\alpha = x$, apply H to each qubit.
2. **"ZZ Phase" Gate.** Now $\sigma_z \otimes \sigma_z$ can be implemented by a known circuit:

   $$\text{CNOT} \;\rightarrow\; R_z(2\,\theta) \;\rightarrow\; \text{CNOT},$$

   which yields $\exp\bigl(-i\,\sigma_z \otimes \sigma_z\,\theta\bigr)$. Set $\theta = \Delta t$ to match $\exp\bigl(-i\,\sigma_z \otimes \sigma_z\,\Delta t\bigr)$.
3. **Undo the Basis Change.** Apply the inverse rotations (Hadamards, etc.) to return to the original computational basis.

Circuit Diagram for $\sigma_x \otimes \sigma_x$. Below is a schematic for $\exp\bigl(-i\,\sigma_x \otimes \sigma_x\,\Delta t\bigr)$ on two qubits ($q0$, $q1$):

```
(q0) ---[H]-----o--------[Rz(2*Delta_t)]-------o------[H]---
                |                              |
(q1) ---[H]-----X------------------------------X------[H]---
```

(b.3) Probability after single Trotter step

Using the involutory nature of the operators ($(\sigma_\alpha \otimes \sigma_\alpha)^2 = I$), we can write for any such operator A where $A^2 = I$:

$$e^{-iAt} = \cos(t)I - i\sin(t)A$$

Let's apply this step by step:

1) First exponential:

$$e^{-i\sigma_x \otimes \sigma_x \Delta t} = \cos(\Delta t)I - i\sin(\Delta t)(\sigma_x \otimes \sigma_x)$$

Acting on $|01\rangle$:

$$|\psi_1\rangle = \cos(\Delta t)|01\rangle - i\sin(\Delta t)|10\rangle$$

2) Second exponential:

$$e^{-i\sigma_y \otimes \sigma_y \Delta t} = \cos(\Delta t)I - i\sin(\Delta t)(\sigma_y \otimes \sigma_y)$$

Acting on $|\psi_1\rangle$:

$$\begin{aligned}
|\psi_2\rangle &= [\cos(\Delta t)I - i\sin(\Delta t)(\sigma_y \otimes \sigma_y)][\cos(\Delta t)|01\rangle - i\sin(\Delta t)|10\rangle] \\
&= \cos^2(\Delta t)|01\rangle - i\sin(\Delta t)\cos(\Delta t)|10\rangle \\
&\quad - i\sin(\Delta t)\cos(\Delta t)|10\rangle - \sin^2(\Delta t)|01\rangle \\
&= [\cos^2(\Delta t) - \sin^2(\Delta t)]|01\rangle - 2i\sin(\Delta t)\cos(\Delta t)|10\rangle \\
&= \cos(2\Delta t)|01\rangle - i\sin(2\Delta t)|10\rangle
\end{aligned}$$

3) Final exponential:

$$e^{-i\sigma_z \otimes \sigma_z \Delta t} = \cos(\Delta t)I - i\sin(\Delta t)(\sigma_z \otimes \sigma_z)$$

Acting on $|\psi_2\rangle$:

$$|\psi(\Delta t)\rangle = e^{i\Delta t}[\cos(2\Delta t)|01\rangle - i\sin(2\Delta t)|10\rangle]$$

Therefore, the probability of measuring the first qubit in state $|0\rangle$ is:

$$\begin{aligned}
P(0) &= |\langle 0|_1|\psi(\Delta t)\rangle|^2 \\
&= |\langle 0|_1[e^{i\Delta t}(\cos(2\Delta t)|01\rangle - i\sin(2\Delta t)|10\rangle)]|^2 \\
&= |e^{i\Delta t}(\langle 0|_1|01\rangle\cos(2\Delta t) - i\langle 0|_1|10\rangle\sin(2\Delta t))|^2
\end{aligned}$$

Now, let's evaluate the matrix elements:

$$\langle 0|_1|01\rangle = 1 \text{ (as this is the amplitude of first qubit being 0)}$$
$$\langle 0|_1|10\rangle = 0 \text{ (as this is the amplitude of first qubit being 1)}$$

Therefore:

$$\begin{aligned}
P(0) &= |e^{i\Delta t}(\cos(2\Delta t) \cdot 1 - i\sin(2\Delta t) \cdot 0)|^2 \\
&= |e^{i\Delta t}\cos(2\Delta t)|^2 \\
&= |e^{i\Delta t}|^2 |\cos(2\Delta t)|^2 \\
&= (1) \cdot \cos^2(2\Delta t) \\
&= \cos^2(2\Delta t)
\end{aligned}$$

where we used that $|e^{i\Delta t}|^2 = 1$ for any real Δt.

Problem 3. *Pasquale Scarlino* (total 20 marks)

Student Name: ______________________________

Section: _______________

Sciper No.: _______________

General questions (6 marks)

Answer to the following questions (try to be concise and straight to the point):

1. What does RWA stand for and what does it consist of doing?

   1 mark

2. Why can't a superconducting resonator be used as a qubit? What essential property do you need and how is it usually implemented in superconducting devices?

   1 mark

3. Here is a plot of a Rabi oscillation. What does it show and how would you use this to calibrate a $\pi/2$ pulse?

   2 mark

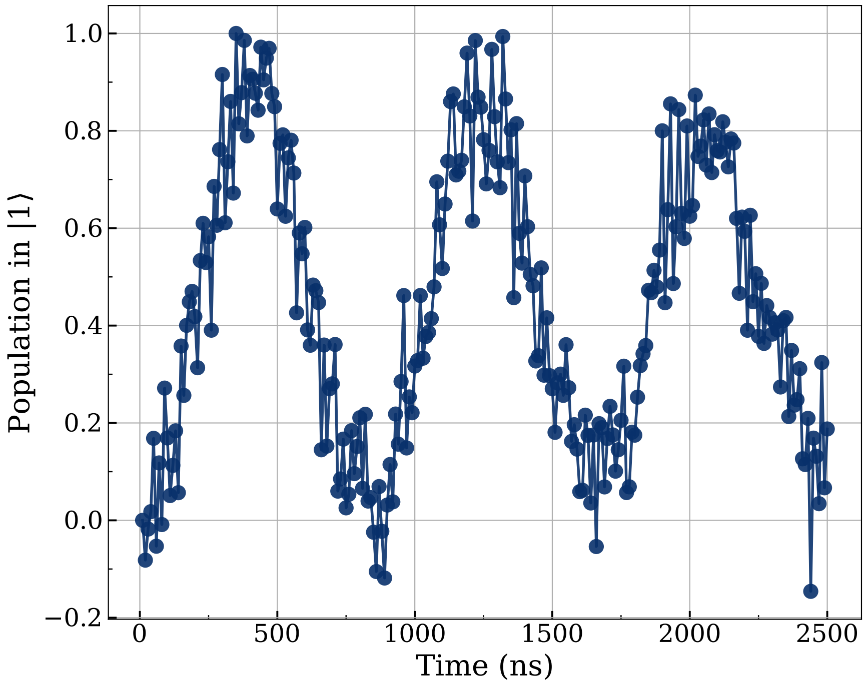


Figure 1: Rabi oscillations (HQC).

4. Explain briefly how a resonator can be used to readout a qubit state.

   2 mark

Transmon Hamiltonian manipulation (14 marks)

In the course, you have extensively seen the Hamiltonian of a transmon qubit (for this exercise, we take $\hbar = 1$):

$$\hat{H} = E_C(\hat{n} - n_g)^2 - E_J \cos(\hat{\delta}). \tag{3}$$

1. What are $\hat{n}$ and $\hat{\delta}$?

   2 marks

2. What is the condition on the charging and Josephson energies to be in the transmon limit?

   2 marks

3. By taking into account properties of the operators, we can express the Hamiltonian in the charge basis:

   $$\hat{H} = \sum_n [E_C(n - n_g)^2 \left|n\right\rangle \left\langle n\right| - \frac{E_J}{2}(\left|n\right\rangle \left\langle n+1\right| + \left|n+1\right\rangle \left\langle n\right|)] \tag{4}$$

   The qubit is generally cooled down to very low temperatures, we can then take the approximation of a two-level system. Rewrite the Hamiltonian in the charge basis in this case.

   2 mark

4. We now start to drive the system with an external line. The charge offset n_g now becomes time dependant: $n_g = n_g^0 + A\cos(\omega_d t + \phi)$. It can be shown by applying a rotating frame at the drive frequency and RWA that the Hamiltonian can be written as:

   $$\hat{H} = \frac{\Delta}{2}\hat{\sigma_z} + \frac{\tilde{A}}{2}(\cos(\phi)\hat{\sigma_x} + \sin(\phi)\hat{\sigma_y}), \tag{5}$$

   where $\tilde{A} = AE_C$.

   What are Δ, A and ϕ?

   2 mark

5. How can you use the Hamiltonian to implement any single qubit gate? What driving parameters (Δ, ϕ) would you choose if you want to implement a π pulse for instance?

   3 mark

6. Use time evolution to determine how much time do you need to turn on this drive in order to complete the π pulse if the initial state is $|\psi(t = 0)\rangle = |0\rangle$?

   3 mark

Solution to problem 3:

General questions (6 marks)

1. RWA stands for "Rotating wave approximation". It consists in neglecting terms in the Hamiltonian when doing a rotating frame. To be more precise, the neglected terms are the ones that oscillate rapidly ($\omega_d + \omega_q$) and can be averaged out considering the time dynamics of the system.

   1 marks

2. A resonator cannot be used as a qubit because it's energy levels are equally spaced (linear, harmonic oscillator). In order to get a two-level system approximation, you need to add non-linearity to the system (anharmonicity) so that the energy levels have different spacings.
   In practice, the nonlinearity is implemented with Josephson Junctions (JJs), a superconducting/isolator/superconducting junction that has a nonlinear phase-current relation.

   1 marks

3. A Rabi oscillation shows the exchange of excitations between two states when a qubit is continuously driven. This can be used to calibrate one-qubit gates as we can see how much time the drive needs to be on in order to cover a certain angle in the Bloch sphere when rotating around an axis.
   In this case, a $\pi/2$ pulse is realised when there is an equal probability to be in $|0\rangle$ and $|1\rangle$ (excited population of 0.5) which is achieved for a 250, 525, 1125... ns drive.

   2 mark

4. The resonator is coupled (usually capacitively) to the qubit. In the dispersive regime, achieved by choosing correctly the detuning between the qubit and resonator frequencies and choosing the right coupling parameters, the excitation of the qubit will occur a shift in frequency of the resonator. By sending a probe tone (readout signal) and reading out the resonator in magnitude (parking ourselves at the resonance frequency of the resonator for example) or in phase (by parking ourselves in between the two freqeuncy shifts), it is possible to determine the qubit state in a QND way.

   2 mark

Transmon Hamiltonian manipulation (14 marks)

1. $\hat{n}$ is the charge operator, representing the number of extra charges that are the transmon superconducting island.
   $\hat{\delta}$ is the phase operator, representing the phase difference across the junction.

2 marks

2. To be in the transmon limit, we want to limit the influence of chareg noise. In other words, we want to "flatten" the energy levels compared to a Cooper Pair Box. This regime is reached for values of charging energy way smaller than josephson energy $E_C << E_J$. This comes at the cost of less anharmonicity.

   2 marks

3. If we are in the two-level system approximation, we only care about the two first energy levels $n = 0, 1$. Plugging into the Hamiltonian:

$$\hat{H} = \sum_n [E_C(n - n_g)^2 |n\rangle \langle n| - \frac{E_J}{2}(|n\rangle \langle n+1| + |n+1\rangle \langle n|)]$$
$$\hat{H} = E_C n_g^2 |0\rangle \langle 0| + E_C(1 - n_g)^2 |1\rangle \langle 1| - \frac{E_J}{2}(|0\rangle \langle 1| + |1\rangle \langle 0|).$$
$$\hat{H} = E_C(n_g^2 |0\rangle \langle 0| + (1 - n_g)^2 |1\rangle \langle 1|) - \frac{E_J}{2}(|0\rangle \langle 1| + |1\rangle \langle 0|).$$

   2 mark

4. Δ is the detuning between the drive and qubit resonance frequencies. A non-zero detuning will then induce rotation around the Z axis of the Bloch sphere.
   A is the drive amplitude, this will determine the speed of rotation around the X and Y axis of the Bloc sphere.
   ϕ is the phase of the drive, it will determine around which axis the rotation is done (X or Y) when the drive in on (A non-zero).

   2 mark

5. By choosing specific drive parameters Δ, ϕ we can do rotations around any axis of the Bloch sphere $(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ and even a combination of rotations. The Bloch sphere represents all the states that a single qubit can reach so this Hamiltonian can do any single qubit gate.
   The simplest parameters to do a π pulse ($|0\rangle$ to $|1\rangle$) is to do a rotation around the X-axis of the bloch sphere so $\Delta = \phi = 0$. The Hamiltonian is then simply:

$$\hat{H} = \frac{\tilde{A}}{2}\hat{\sigma}_x.$$

   3 mark

6. The time evolution of an arbitrary state under a time-independent Hamiltonian is:

$$|\psi(t)\rangle = e^{-i\hat{H}t} |\psi(0)\rangle .$$

In our case:

$$|\psi(t)\rangle = e^{-i\frac{\tilde{A}}{2}\hat{\sigma}_x t}\,|0\rangle\,.$$

By using properties from Pauli matrices:

$$e^{\pm i\frac{\theta}{2}(n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z)} = \cos(\frac{\theta}{2})\mathbb{1} \pm i\sin(\frac{\theta}{2})(n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z),$$

we obtain:

$$\begin{aligned}
|\psi(t)\rangle &= [\cos(\frac{\tilde{A}}{2}t)\mathbb{1} - i\sin(\frac{\tilde{A}}{2}t)\hat{\sigma}_x]\,|0\rangle \\
|\psi(t)\rangle &= [\cos(\frac{\tilde{A}}{2}t)\mathbb{1} - i\sin(\frac{\tilde{A}}{2}t)(|0\rangle\langle 1| + |1\rangle\langle 0|)]\,|0\rangle \\
|\psi(t)\rangle &= \cos(\frac{\tilde{A}}{2}t)\,|0\rangle - i\sin(\frac{\tilde{A}}{2}t)\,|1\rangle
\end{aligned}$$

As expected we get a superposition of state $|0\rangle$ and $|1\rangle$ with an oscillatory behavior. To complete a π pulse, we want to find a time t_f for which the state becomes $|\psi(t_f)\rangle = |1\rangle$, so:

$$\begin{aligned}
\frac{\tilde{A}}{2}t &= \frac{\pi}{2} \\
\Rightarrow t &= \frac{\pi}{\tilde{A}}.
\end{aligned}$$

One can notice that the harder we drive the system, the less time we need to wait with the drive on and the faster the gate.

3 mark

Problem 4. *Adrian Ionescu* (total 20 marks)

Student Name: ____________________________

Section: ______________

Sciper No.: ______________

Please indicate the correct answers for each of the following questions by circling them. Multiple correct answers may apply.

- +1 point will be awarded for each correct answer.
- -1 point will be deducted for each incorrect answer.
- No selection will result in 0 points.

Note: The overall score for each question will be capped at zero to prevent excessive penalties that could unduly affect the total exam score in the event of numerous incorrect answers.

Question 1 (Ionescu):

Select the correct statements regarding the semiconducting Tunnel Field-Effect Transistors (Tunnel FETs), which utilize quantum mechanical band-to-band tunneling as the conduction mechanism, from the list below:

1. Tunnel FETs inherit several performance-enhancing technological features from MOSFETs, called technology boosters. These include: operation at cryogenic temperatures, gate length scaling, and multi-gate control (e.g., FinFET and double-gate architectures).
2. At low gate voltages, the subthreshold slope of a Tunnel FET is non-constant and dependent on the gate voltage.
3. Due to their low subthreshold slope at low voltages, Tunnel FETs can be used to design more energy-efficient analog amplifiers with reduced temperature sensitivity.
4. The geometry of the tunneling barrier in Tunnel FETs is identical to the one of Single Electron Transistors; only the technological implementation differs.
5. Homojunction Tunnel FETs can, in principle, achieve higher Ion/Ioff ration than heterostructure Tunnel FETs.
6. Trap-Assisted Tunneling (TAT) significantly reduces the subthreshold slope, making Tunnel FETs promising candidates for low-voltage electronic circuits.
7. Trap-Assisted Tunneling (TAT) is a phenomenon that is dependent of temperature.
8. The off-state current (Ioff) of homojunction Tunnel FETs is expected to increase at cryogenic temperatures (sub-77K) compared to room temperature (300K).

9. In an ideal Tunnel FET, where band-to-band tunneling predominates, the subthreshold slope is expected to be much smaller than the 60 mV/decade limit observed in MOSFETs at room temperature (300K).
10. The Zener diode utilizes the same physical mechanism as the Tunnel FET for electrical conduction.

Question 2 (Ionescu):

Single Electron Transistors (SETs) are three-terminal devices that exploit discrete charge tunneling, with a conductive nanodot as the central island. This island is separated from the source and drain electrodes by tunneling barriers. Select the correct statements from the list below that accurately describe the features, operation, and technological implementation of SETs.

1. When using a metallic central island with a 2 nm radius, surrounded by three metal electrodes (gate, source, and drain) with significantly larger radii, the resulting SET is expected to exhibit effective Coulomb blockade at T = 200 K, making it suitable for constructing a SET inverter.

2. A parasitic background charge effect primarily influences the levels of the drain current (Ion) at same bias in Single Electron Transistors, but does not affect the threshold voltages in the ID-VG characteristics.

3. The SET device exhibits multiple threshold voltages with respect to both the gate and drain voltages, which are a direct result of the Coulomb blockade (or gap) effect.

4. In the current state of the art, the intrinsic device operating frequency range of SETs —determining the speed of the device— can be typically higher than GHz in optimal cases, despite their low on-current (Ion).

5. The orthodox theory of SETs is based on the Heisenberg uncertainty principle; however, aside from the constraints on the size of the central nano-island necessary for operation at a given temperature, there are no quantitative constraints on other SET parameters, such as tunneling capacitances or resistances.

6. The SET exhibits a much steeper subthreshold slope than a MOSFET at any given temperature.

7. An inverter based on two SETs consumes both static and dynamic power during the transition between logic states.

8. In a SET-MOS hybrid device, combining SET and MOSFET components, the periodicity of the ID-VG transfer characteristics is predominantly governed by the SET gate capacitance.

9. It is possible to design a device that, at certain gate voltages, operates similarly to an SET, while at other gate voltages, behaves like a MOSFET.

10. The SET devices are used as electrical readouts of the states of spin-based qubits.

Solution to problem 4:

Adrian type your solution here

The correct answers for Question 1 are: 2, 3, 5, 7, 10

The correct answers for Question 2 are: 1, 3, 4, 7, 8, 9, 10

Problem 5. *Edoardo Charbon* (total 20 marks)

Student Name: ___________________________

Section: _____________

Sciper No.: _____________

1. What do you control in a qubit? How?
   1 marks
2. Give an example of the electrical pulses used to control a transmon.
   1 marks
3. How is the envelope of the pulses used in the control of a qubit impacting scalability?
   2 marks
4. In the readout of qubits, what is the first component you find? What is its purpose?
   2 marks
5. What are the 3 most critical performance measures in an cryo-LNA? Can you give a range of values for these performance measures generally found in a qubit readout?
   3 marks
6. Draw a simple schematic of a differential LNA. Indicate active and passive components. Indicate a feedback loop if you need one. Is the amplifier inverting or non-inverting?
   3 marks
7. Find the gain of the LNA at low frequencies in symbols. What is the 3-dB bandwidth of the gain in symbols? What happens to the gain after this point?
   3 marks
8. How do you compute the input-referred noise in terms of noise-equivalent temperature? What is its meaning?
   2 marks
9. What is the minimum supply voltage required in a digital circuit at 4K? What are the trade-offs between power and bandwidth in logic circuits at room temperature? How does this change at cryogenic temperatures?
   3 marks

Solution to problem 5:

Edoardo type solution here: