EXERCISE 1

1. $$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}$$

we want to write it in the form:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

let's rewrite it as:

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C \frac{1}{LC} \hat{\phi}^2$$

and now we can identify:

$$\hat{Q} \leftrightarrow \hat{p}$$
$$\hat{\Phi} \leftrightarrow \hat{x}$$
$$C \leftrightarrow m$$
$$\frac{1}{\sqrt{LC}} \leftrightarrow \omega$$

2. For the mechanical oscillator:

$$x_{ZPF} = \sqrt{\frac{\hbar}{2m\omega}} \qquad p_{ZPF} = \sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2x_{ZPF}}$$

so for the LC oscillator:

$$\phi_{ZPF} = \sqrt{\frac{\hbar\sqrt{LC}}{2C}} = \sqrt{\frac{\hbar}{2}}\sqrt{\frac{L}{C}}$$

$$Q_{ZPF} = \sqrt{\frac{\hbar C}{2\sqrt{LC}}} = \sqrt{\frac{\hbar}{2}}\sqrt{\frac{C}{L}}$$

EXERCISE 2

1 $-E_J \cos(\hat{\delta}) \approx -E_J\left(\hat{\mathbb{1}} - \frac{1}{2}\hat{\delta}^2\right)$ (Taylor expansion)

$$\Rightarrow \hat{H} \approx 4E_C \hat{N}^2 \underbrace{- E_J}_{\text{constant term, can be neglected}} + \frac{E_J}{2}\hat{\delta}^2 = 4E_C\hat{N}^2 + \frac{E_J}{2}\hat{\delta}^2 + \text{const}$$

Now we replace $\hat{N} = \frac{\hat{Q}}{2e}$, $\hat{\delta} = 2\pi \frac{\hat{\Phi}}{\Phi_0}$

$$\hat{H} \approx \frac{E_C}{e^2}\hat{Q}^2 + \frac{E_J}{2}\cdot\frac{4\pi^2}{\Phi_0^2}\hat{\Phi}^2$$

2 Comparing with the LC oscillator we can see that

$$\text{"}C\text{"} \simeq \frac{e^2}{2E_C}, \qquad \text{"}L\text{"} \simeq \frac{\Phi_0^2}{4\pi^2 E_J}$$

So:

$$\omega = \frac{1}{\sqrt{\frac{e^2\Phi_0^2}{8\pi^2 E_C E_J}}} = \frac{\sqrt{8E_C E_J}}{\sqrt{\underbrace{\frac{\cancel{e^2}}{\pi^2}\frac{h^2}{4\cancel{e^2}}}_{\hbar^2}}} = \frac{1}{\hbar}\sqrt{8E_C E_J}$$

So

$$\hat{H} \approx \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \quad \text{with} \quad \hbar\omega = \sqrt{8E_C E_J}$$

3 $-E_J\cos(\hat{\delta}) \approx -E_J\left(\hat{\mathbb{1}} - \frac{1}{2}\hat{\delta}^2 + \frac{1}{24}\hat{\delta}^4\right)$

$$\Rightarrow H = 4E_C\hat{N}^2 + \frac{E_J}{2}\hat{\delta}^2 - \frac{E_J}{24}\hat{\delta}^4$$

$$= 4E_C\left(\frac{\hat{Q}}{2e}\right)^2 + \frac{E_J}{2}\left(2\pi\frac{\hat{\Phi}}{\Phi_0}\right)^2 - \frac{E_J}{24}\left(2\pi\frac{\hat{\Phi}}{\Phi_0}\right)^4 =$$

$$= \frac{E_C}{e^2}\hat{Q}^2 + \frac{E_J\cdot 4\pi^2}{2\Phi_0^2}\hat{\Phi}^2 - \frac{E_J\cdot(2\pi)^4}{24\ \Phi_0^4}\hat{\Phi}^4$$

4 We have already found before that the first two terms are the one of a perfect Q.H.O., so in second quantization they give $\hbar\omega(\hat a^+\hat a + \frac{1}{2})$. We only have to write the third term in second quantization:

$$\hat\phi^4 = \phi_{ZPF}^4 \, (\hat a + \hat a^+)^4$$

- $$\phi_{ZPF}^4 = \left(\sqrt{\frac{\hbar}{2}\sqrt{\frac{\text{"}L\text{"}}{\text{"}C\text{"}}}}\right)^4 = \frac{\hbar^2}{4}\,\frac{\text{"}L\text{"}}{\text{"}C\text{"}} = \frac{\hbar^2}{{}_2\cancel{4}}\,\frac{\phi_0^2}{(2\pi)^2 E_J}\cdot\frac{\cancel{2}E_C}{e^2} =$$

$$= \frac{1}{2}\,\overset{\phi_0^2}{\frac{\hbar^2}{4e^2}}\,\frac{4\,\phi_0^2}{(2\pi)^4}\,\frac{E_C}{E_J} = \frac{\phi_0^4}{(2\pi)^4}\,\frac{2E_C}{E_J}$$

- $$(\hat a + \hat a^+)^4 = (a+a^+)^2(a+a^+)^2 =$$

$$= (a^2 + a^{+2} + aa^+ + a^+a)(a^2 + a^{+2} + aa^+ + a^+a) =$$

$$= (\cancel{a^4} + a^2a^{+2} + \cancel{a^3a^+} + \cancel{a^2a^+a} +$$

$$+ a^{+2}a^2 + \cancel{a^{+4}} + \cancel{a^{+2}aa^+} + \cancel{a^{+3}a} +$$

$$+ \cancel{aa^+a^2} + \cancel{aa^{+3}} + aa^+aa^+ + aa^+a^+a +$$

$$+ \cancel{a^+a^3} + \cancel{a^+aa^{+2}} + a^+aaa^+ + a^+aa^+a)$$

We can neglect all terms where the power of a and a^+ are unequal

$$\approx a^2a^{+2} + a^{+2}a^2 + aa^+aa^+ + aa^+a^+a + a^+aaa^+ + a^+aa^+a$$

Let's use $[a,a^+] = 1 \Rightarrow aa^+ - a^+a = 1$

$$[a^2, a^{+2}] = a\,[a, a^{+2}] + [a, a^{+2}]\,a =$$

$$= a\,[a,a^+]\,a^+ + aa^+\,[a,a^+] + a^+[a,a^+]\,a + [a,a^+]\,a^+a$$

$$= aa^+ + aa^+ + a^+a + a^+a = 2aa^+ + 2a^+a = 2 + 4a^+a$$

$$\Rightarrow a^2a^{+2} = a^{+2}a^2 + 4a^+a + \text{const}$$

$$aa^+aa^+ = (a^+a+1)(a^+a+1) = a^+aa^+a + 2a^+a + \text{const}$$
$$= a^+(a^+a+1)a + 2a^+a = a^{+2}a^2 + 3a^+a$$

$$aa^+a^+a = (a^+a+1)a^+a = a^+aa^+a + a^+a =$$
$$= a^+(a^+a+1)a + a^+a =$$
$$= a^{+2}a^2 + 2a^+a$$

$$a^+aaa^+ = a^+a(a^+a+1) = a^+aa^+a + a^+a =$$
$$= a^{+2}a^2 + 2a^+a$$

$$a^+aa^+a = a^{+2}a^2 + a^+a$$

Summing everything up:

$$(a+a^+)^4 = 6a^{+2}a^2 + 12a^+a = 6(a^{+2}a^2 + 2a^+a)$$

We can finally go back to the Hamiltonian:

$$H = \underbrace{\frac{E_C}{e^2}\hat{Q}^2 + \frac{4\pi^2}{2\Phi_0^2}E_J\hat{\phi}^2}_{} - \frac{E_J\cdot 16\pi^4}{24\,\Phi_0^4}\underbrace{\hat{\phi}^4}_{}$$

$$= \overbrace{\hbar\omega\left(a^+a+\frac{1}{2}\right)}^{} - \frac{\cancel{(2\pi)^4}E_J}{\cancel{24}_{4}\,\cancel{\Phi_0^4}}\overbrace{\frac{\cancel{\Phi_0^4}}{\cancel{(2\pi)^4}}\,2\frac{E_C}{E_J}\cdot\cancel{6}\left(a^{+2}a^2 + 2a^+a\right)}^{} =$$

$$= \hbar\omega\left(a^+a+\frac{1}{2}\right) - \frac{E_C}{2}\left(a^{+2}a^2 + 2a^+a\right) + \text{const}$$

this result in a shift of ω

$$= \hbar\omega'\left(a^+a+\frac{1}{2}\right) - \frac{E_C}{2}\left(a^{+2}a^2\right)$$

with $\hbar\omega' = \sqrt{8E_CE_J} - E_C \approx \sqrt{8E_CE_J}$ if $E_C \ll E_J$