Introduction to quantum science and technology: qubit platforms
Session 3

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Second quantization of superconducting circuits

Introduction

In this exercise session we look at how to describe superconducting circuits using second quantization. You probably are already familiar with the concept of the *ladder operators*, usually denoted by $\hat{a}$ and $\hat{a}^\dagger$, that are usually introduced in quantum mechanics courses when dealing with harmonic oscillators.

Let us quickly recall the most important concepts of a mechanical quantum harmonic oscillator, which will be useful when dealing with superconducting circuits. A mechanical quantum harmonic oscillator is described by the Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{1}$$

one can introduce the conjugate operators:

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + i\frac{\hat{p}}{\sqrt{\hbar m\omega}}\right), \qquad \hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} - i\frac{\hat{p}}{\sqrt{\hbar m\omega}}\right), \tag{2}$$

such that $\hat{x}$ and $\hat{p}$ can be rewritten as:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) = x_0(\hat{a} + \hat{a}^\dagger), \tag{3}$$

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger) = -ip_0(\hat{a} - \hat{a}^\dagger), \tag{4}$$

with

$$x_0 = \sqrt{\frac{\hbar}{2m\omega}}, \tag{5}$$

$$p_0 = \sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2x_0}. \tag{6}$$

By doing this substitution, one can rewrite Hamiltonian 1 as:

$$H = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}). \tag{7}$$

In can be shown that the eigenvalues of $\hat{a}^\dagger\hat{a}$ can only take non-negative integer values $n = 0, 1, 2, \ldots,$ and $\hat{a}^\dagger\hat{a}$ can be interpreted as the operator counting the excitations of the system. Hence the eigenvalues of the Hamiltonian (i.e. its eigenenergies) are:

$$E_n = \hbar\omega(n + \frac{1}{2}). \tag{8}$$

Moreover, it can be shown that the operator $\hat{a}$ alone acts by removing one excitation from the system, while $\hat{a}^\dagger$ alone acts by adding one excitation to the system. The concept of *second quantization* consists exactly in describing a quantum harmonic oscillator in terms of its excitations using the ladder operators $\hat{a}$ and $\hat{a}^\dagger$. Finally, the quantities x_0 and p_0 represent the fluctuations (or standard deviations) of the operators $\hat{x}$ and $\hat{p}$ on the ground state of the Hamiltonian, i.e. with $n = 0$. Therefore they are usually called *zero-point fluctuations*.

Exercise 1: Second quantization of an LC circuit

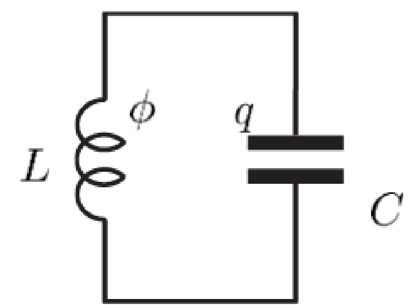

Figure 1: LC circuit

Let us consider a superconducting LC circuit like the one represented in figure 1. Recall that its Hamiltonian, written using the conjugate variables Q and Φ, is:

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} \tag{9}$$

1. Show that Hamiltonian 9 can be rewritten exactly in the form of a quantum harmonic oscillator (QHO) Hamiltonian like Eq. 1. Write the correspondence between $\{\hat{p}, \hat{x}, m, \omega\}$ and $\{\hat{Q}, \hat{\Phi}, C, L\}$.

2. After doing the point above, taking inspiration from the QHO write the charge operator $\hat{Q}$ and the flux operator $\hat{\Phi}$ using the ladder operators $\hat{a}, \hat{a}^\dagger$, as:

$$\hat{Q} = -iQ_{\text{ZPF}}(\hat{a} - \hat{a}^\dagger) \tag{10}$$

$$\hat{\Phi} = \Phi_{\text{ZPF}}(\hat{a} + \hat{a}^\dagger) \tag{11}$$

   and find the expression of their zero-point fluctuations Q_{ZPF} and Φ_{ZPF}.

Exercise 2: Second quantization of the CPB in the transmon regime

Let us now recall from the lectures and from the last exercise class the Hamiltonian of a Cooper Pair Box:

$$\hat{H} = 4E_C(\hat{N} - n_g)^2 - E_J \cos\left(\hat{\delta}\right) \tag{12}$$

where $\hat{N} = \hat{Q}/(2e)$ and $\hat{\delta} = 2\pi\hat{\Phi}/\Phi_0$ (Φ_0 is the superconducting flux quantum). We now focus on the *transmon regime*, where we take $E_J \gg E_C$ and neglect the offset charge n_g. We can see that the potential term of the Hamiltonian is cosinusoidal in the flux, instead of quadratic in the flux (as was for the LC circuit). However, we can expand the cosine around $\hat{\delta} = 0$ to retrieve back the shape of a QHO.

1. Expand the term in $\hat{\delta}$ in the Hamiltonian of the CPB up to the second order in $\hat{\delta}$, and then rewrite the Hamiltonian in terms of $\hat{Q}$ and $\hat{\Phi}$, neglecting the offset charge.

2. Write now the resulting Hamiltonian in second quantization. What is ω in terms of E_C and E_J?

Now we want to also take into account the deviation of the Josephson junction from a perfectly linear inductor, which until now is not captured by expanding the cosine only up to second order.

3. Expand the term $-E_J \cos\left(\hat{\delta}\right)$ around $\hat{\delta} = 0$ up to the fourth order in $\hat{\delta}$. Then write the full Hamiltonian in terms of $\hat{Q}$ and $\hat{\Phi}$

4. Finally, apply the second quantization to rewrite the resulting Hamiltonian in terms of $\hat{a}$ and $\hat{a}^\dagger$. You should find a Hamiltonian in the form:

$$\hat{H} = \hbar\omega(\hat{a}^\dagger a + \frac{1}{2}) - \frac{\hbar\alpha}{2}\,\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}. \tag{13}$$

   Identify the expressions for ω and α, and then try to explain what is the physical meaning of $\hbar\alpha$.