EXERCISE 1

1 $H = \frac{\hbar\omega_0}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right) + \hbar\Omega\cos(\omega_d t - \phi)\left(|g\rangle\langle e| + |e\rangle\langle g|\right) =$

$$= -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\Omega\cos(\omega_d t - \phi)\,\sigma_x =$$

$$= \begin{pmatrix} -\hbar\omega_0/2 & \hbar\Omega\cos(\omega_d t - \phi) \\ \hbar\Omega\cos(\omega_d t - \phi) & +\hbar\omega_0/2 \end{pmatrix}$$

The diagonal terms represent the energy of the uncoupled states $|g\rangle$ and $|e\rangle$, while the drive off-diagonal terms make the system transition $|g\rangle \leftrightarrow |e\rangle$.

2 $U = e^{-i\frac{\omega_d t}{2}\sigma_z} = e^{-i\frac{\omega_d t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)}$

$U H_0 U^\dagger = e^{-i\frac{\omega_d t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)}\ \frac{\hbar\omega_0}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right) \times$

since U is diagonal, we can do direct operator product

$$\times\ e^{+i\frac{\omega_d t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)} =$$

$$= \frac{\hbar\omega_0}{2}\, e^{-i\frac{\omega_d t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)}\,|e\rangle\langle e|\, e^{+i\frac{\omega_d t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)} +$$

$$- \frac{\hbar\omega_0}{2}\, e^{-i\frac{\omega_d t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)}\,|g\rangle\langle g|\, e^{+i\frac{\omega_d t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)} =$$

$$= \frac{\hbar\omega_0}{2}\left(\cancel{e^{+i\frac{\omega_d t}{2}}}\,|e\rangle\langle e|\,\cancel{e^{-i\frac{\omega_d t}{2}}} - \cancel{e^{-i\frac{\omega_d t}{2}}}\,|g\rangle\langle g|\,\cancel{e^{+i\frac{\omega_d t}{2}}}\right) =$$

$$= \frac{\hbar\omega_0}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right) = H_0$$

$U H_d U^\dagger = e^{-i\frac{\omega_d t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)}\ \hbar\Omega\cos(\omega_d t - \phi)\left(|g\rangle\langle e| + |e\rangle\langle g|\right) \times$

$$\times\ e^{+i\frac{\omega_d t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)} =$$

$$= \hbar\Omega\cos(\omega_d t - \phi)\Big[\, e^{-i\frac{\omega_d t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)}\,|g\rangle\langle e|\, e^{+i\frac{\omega_d t}{2}(|g\rangle\langle g - |e\rangle\langle e|)} +$$

$$+ e^{-i\frac{\omega_d t}{2}(|g\rangle\langle g|-|e\rangle\langle e|)} |e\rangle\langle g| e^{+i\frac{\omega_d t}{2}(|g\rangle\langle g|-|e\rangle\langle e|)}\Big]$$

$$= \hbar\Omega\cos(\omega_d t-\phi)\left[e^{-i\frac{\omega_d t}{2}} |g\rangle\langle e| e^{-i\frac{\omega_d t}{2}} + e^{+i\frac{\omega_d t}{2}} |e\rangle\langle g| e^{+i\frac{\omega_d t}{2}}\right] =$$

$$= \frac{\hbar\Omega}{2}\left(e^{i\omega_d t - i\phi} + e^{-i\omega_d t + i\phi}\right)\left(e^{-i\omega_d t}|g\rangle\langle e| + e^{+i\omega_d t}|e\rangle\langle g|\right) =$$

$$= \frac{\hbar\Omega}{2}\left[\left(e^{-i\phi} + e^{-2i\omega_d t + i\phi}\right)|g\rangle\langle e| + \left(e^{+i\phi} + e^{2i\omega_d t - i\phi}\right)|e\rangle\langle g|\right]$$

$$i\hbar \frac{dU}{dt}U^\dagger = i\hbar\frac{d}{dt}\left(e^{-i\frac{\omega_d t}{2}(|g\rangle\langle g|-|e\rangle\langle e|)}\right) e^{+i\frac{\omega_d t}{2}(|g\rangle\langle g|-|e\rangle\langle e|)} =$$

$$= i\hbar\left(-i\frac{\omega_d t}{2}(|g\rangle\langle g|-|e\rangle\langle e|)\right) =$$

$$= +\frac{\hbar\omega_d t}{2}\left(|g\rangle\langle g|-|e\rangle\langle e|\right) = -\frac{\hbar\omega_d}{2}\left(|e\rangle\langle e|-|g\rangle\langle g|\right)$$

Let's combine all the pieces together:

$$H' = \frac{\hbar(\omega_0-\omega_d)}{2}\left(|e\rangle\langle e|-|g\rangle\langle g|\right) +$$

$$+\frac{\hbar\Omega}{2}\left[\left(e^{-i\phi} + e^{-2i\omega_d t + i\phi}\right)|g\rangle\langle e| + \left(e^{+i\phi} + e^{2i\omega_d t - i\phi}\right)|e\rangle\langle g|\right]$$

3 RWA: discard terms with $e^{\pm(2i\omega_d t - i\phi)}$

$$\Rightarrow H' = \frac{\hbar(\omega_0-\omega_d)}{2}\left(|e\rangle\langle e|-|g\rangle\langle g|\right) + \frac{\hbar\Omega}{2}\left(e^{-i\phi}|g\rangle\langle e| + e^{+i\phi}|e\rangle\langle g|\right)$$

$$= \begin{pmatrix} -\hbar(\omega_0-\omega_d)/2 & \frac{\hbar\Omega}{2}e^{-i\phi} \\ \frac{\hbar\Omega}{2}e^{+i\phi} & +\hbar(\omega_0-\omega_d)/2 \end{pmatrix}$$

If we define $\Delta = \omega_0 - \omega_d$:

$$\boxed{H' = \frac{\hbar}{2}\begin{pmatrix} -\Delta & \Omega e^{-i\phi} \\ \Omega e^{i\phi} & +\Delta \end{pmatrix}}$$

4 To diagonalize, we must solve the secular equation:

identity matrix

$$\det\left(H' - E\mathbb{1}\right) = 0$$

$$\Rightarrow \det\begin{pmatrix} -\frac{\hbar\Delta}{2} - E & \frac{\hbar\Omega}{2} e^{-i\varphi} \\ \frac{\hbar\Omega}{2} e^{i\varphi} & \frac{\hbar\Delta}{2} - E \end{pmatrix} = 0$$

$$\Rightarrow -\left(\frac{\hbar\Delta}{2} + E\right)\left(\frac{\hbar\Delta}{2} - E\right) - \left(\frac{\hbar\Omega}{2}\right)^2 = 0$$

$$\Rightarrow -\left(\left(\frac{\hbar\Delta}{2}\right)^2 - E^2\right) = \left(\frac{\hbar\Omega}{2}\right)^2$$

$$\Rightarrow E^2 = \left(\frac{\hbar\Delta}{2}\right)^2 + \left(\frac{\hbar\Omega}{2}\right)^2$$

$$\Rightarrow \boxed{E = \pm \frac{\hbar}{2}\sqrt{\Delta^2 + \Omega^2}}$$

Let's call $\delta = \sqrt{\Delta^2 + \Omega^2}$

$$\Rightarrow E = \pm \frac{\hbar\delta}{2}$$

Now let's find the eigenvectors. Let them be:

$$E_+ \rightarrow |+\rangle = \alpha_+|g\rangle + \beta_+|e\rangle$$

$$E_- \rightarrow |-\rangle = \alpha_-|g\rangle + \beta_-|e\rangle$$

$|+\rangle$ $\quad H|+\rangle = E_+|+\rangle$

$$\Rightarrow \quad \cancel{\frac{\hbar}{2}}\begin{pmatrix} -\Delta & \Omega e^{-i\varphi} \\ \Omega e^{i\varphi} & \Delta \end{pmatrix}\begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix} = \frac{\cancel{\hbar}\delta}{\cancel{2}}\begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix}$$

$$\Rightarrow \begin{cases} -\Delta\alpha_+ + \Omega e^{-i\varphi}\beta_+ = \delta\,\alpha_+ \\ \Omega e^{i\varphi}\alpha_+ + \Delta\beta_+ = \delta\,\beta_+ \end{cases}$$

Remember that the two equations are linearly dependent, so we cannot solve for α_+ and β_+ independently. We can freely choose one of them, e.g. $\alpha_+ = 1$. Then from the first equation we get β_+:

$$-\Delta + \Omega e^{-i\varphi}\beta_+ = \delta \quad \Rightarrow \quad \beta_+ = \frac{\delta+\Delta}{\Omega e^{-i\varphi}}$$

$$\Rightarrow |+\rangle = \begin{pmatrix} 1 \\ \frac{\delta+\Delta}{\Omega e^{-i\varphi}} \end{pmatrix} \underset{\text{to}}{\overset{\text{proportional}}{\propto}} \begin{pmatrix} \Omega e^{-i\varphi} \\ \delta+\Delta \end{pmatrix}$$

We can normalize it

$$\langle\tilde{+}|\tilde{+}\rangle = (\delta+\Delta)^2 + \Omega^2$$

$$|+\rangle_{norm} = \frac{1}{\sqrt{(\delta+\Delta)^2+\Omega^2}}\begin{pmatrix} \Omega e^{-i\varphi} \\ \delta+\Delta \end{pmatrix}$$

$|-\rangle$ $\quad H|-\rangle = E_-|-\rangle$

$$\Rightarrow \cancel{\frac{\hbar}{2}}\begin{pmatrix} -\Delta & \Omega e^{-i\varphi} \\ \Omega e^{i\varphi} & \Delta \end{pmatrix}\begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix} = -\frac{\cancel{\hbar}\delta}{\cancel{2}}\begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix}$$

$$\Rightarrow \begin{cases} -\Delta\alpha_- + \Omega e^{-i\varphi}\beta_- = -\delta\alpha_- & (1) \\ \Omega e^{i\varphi}\alpha_- + \Delta\beta_- = -\delta\beta_- & (2) \end{cases}$$

$$\alpha_- = 1 \overset{(2)}{\Longrightarrow} \Omega e^{i\varphi} + \Delta\beta_- = -\delta\beta_- \Rightarrow \beta_- = -\frac{\Omega e^{i\varphi}}{\delta+\Delta}$$

$$|\tilde{-}\rangle = \begin{pmatrix} 1 \\ -\frac{\Omega e^{i\varphi}}{\delta+\Delta} \end{pmatrix} \propto \begin{pmatrix} \delta+\Delta \\ -\Omega e^{i\varphi} \end{pmatrix}$$

$$\langle\tilde{-}|\tilde{-}\rangle = (\delta+\Delta)^2 + \Omega^2 \Rightarrow |-\rangle_{norm} = \frac{1}{\sqrt{(\delta+\Delta)^2+\Omega^2}}\begin{pmatrix} \delta+\Delta \\ -\Omega e^{i\varphi} \end{pmatrix}$$

$$\boxed{\begin{aligned} |+\rangle_{norm} &= \frac{1}{\sqrt{(\delta+\Delta)^2+\Omega^2}}\begin{pmatrix} \Omega e^{-i\varphi} \\ \delta+\Delta \end{pmatrix} \\ |-\rangle_{norm} &= \frac{1}{\sqrt{(\delta+\Delta)^2+\Omega^2}}\begin{pmatrix} \delta+\Delta \\ -\Omega e^{i\varphi} \end{pmatrix} \end{aligned}}$$

<u>Note</u>: you can get these eigenstates written in a different form depending on whether you use the first or the second equation and whether you use $\alpha_\pm = 1$ or $\beta_\pm = 1$ to solve.

5 $\quad |\psi(t)\rangle = e^{-i\frac{H'}{\hbar}t}|\psi_0\rangle =$

$$= e^{-i\frac{H'}{\hbar}t}|g\rangle =$$

Write $|g\rangle$ on the $|+\rangle, |-\rangle$ basis:
$|g\rangle = |+\rangle\langle+|g\rangle + |-\rangle\langle-|g\rangle$

$$= e^{-i\frac{H'}{\hbar}t}\left(|+\rangle\langle+|g\rangle + |-\rangle\langle-|g\rangle\right) =$$

$$= e^{-iE_+t/\hbar}\,|+\rangle\langle+|g\rangle + e^{-iE_-t/\hbar}\,|-\rangle\langle-|g\rangle =$$

$$= e^{i\delta t/2}\,|+\rangle\langle+|g\rangle + e^{-i\delta t/2}\,|-\rangle\langle-|g\rangle$$

$$= e^{i\delta t/2}\frac{\Omega e^{i\varphi}}{\sqrt{(\delta+\Delta)^2+\Omega^2}}|+\rangle + e^{-i\delta t/2}\frac{\delta+\Delta}{\sqrt{(\delta+\Delta)^2+\Omega^2}}|-\rangle$$

$$P_e(t) = |\langle e|\psi(t)\rangle|^2 =$$

$$= \left| e^{i\delta t/2}\frac{\Omega e^{i\varphi}}{\sqrt{(\delta+\Delta)^2+\Omega^2}}\langle e|+\rangle + e^{-i\delta t/2}\frac{\delta+\Delta}{\sqrt{(\delta+\Delta)^2+\Omega^2}}\langle e|-\rangle\right|^2 =$$

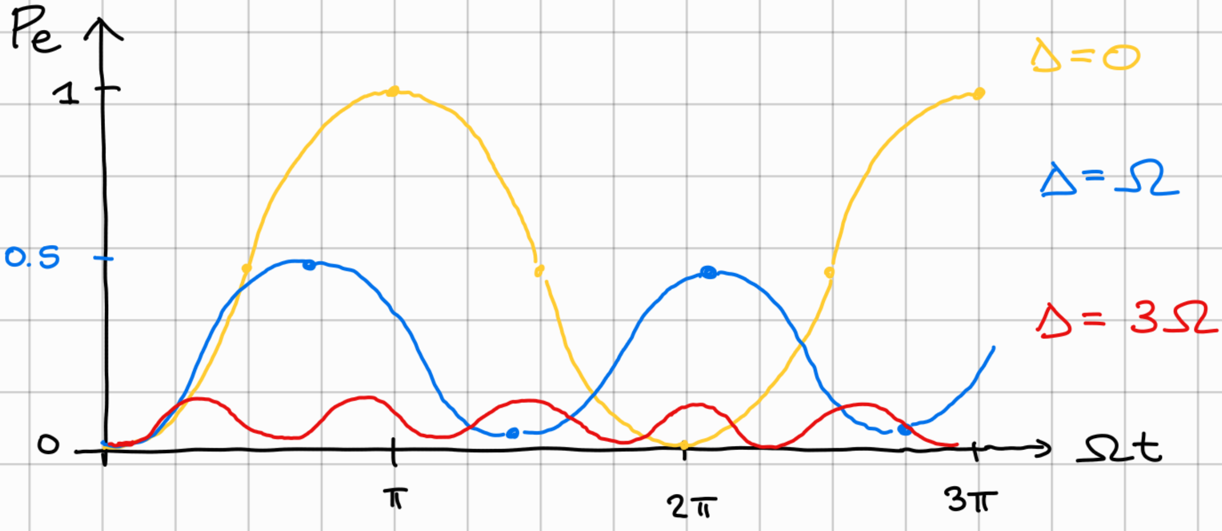
$$= \left| e^{i\delta t/2}\frac{\Omega \cancel{e^{i\varphi}}(\delta+\Delta)}{(\delta+\Delta)^2+\Omega^2} + e^{-i\delta t/2}\frac{(\delta+\Delta)(-\Omega\cancel{e^{i\varphi}})}{(\delta+\Delta)^2+\Omega^2}\right|^2 =$$

$$= \left(\frac{\Omega(\delta+\Delta)}{(\delta+\Delta)^2+\Omega^2}\right)^2 \left|e^{i\delta t/2} - e^{-i\delta t/2}\right|^2 =$$

$$= \left(\frac{\Omega(\delta+\Delta)}{\delta^2+\underline{\Delta^2}+2\delta\Delta+\underbrace{\Omega^2}_{\delta^2}}\right)^2 4\sin^2\left(\frac{\delta t}{2}\right) =$$

$$= \left(\frac{\Omega\cancel{(\delta+\Delta)}}{\cancel{2}\delta\cancel{(\delta+\Delta)}}\right)^2 \cdot \cancel{4}\sin^2\left(\frac{\delta t}{2}\right) =$$

$$\boxed{= \frac{\Omega^2}{\Delta^2+\Omega^2}\sin^2\left(\frac{\delta t}{2}\right)}$$



7 If $\Delta = 0$, from the above formula we obtain that $P_e(t) = \sin^2\left(\frac{\Omega t}{2}\right)$ If $\Omega t = \pi$, then $P_e(t = \pi/\Omega) = 1$

$\Rightarrow$ population transfer from $|g\rangle$ to $|e\rangle$

If $\Omega t = \frac{\pi}{2}$, then $P_e(t = \pi/2\Omega) = 1/2$. Furthermore, we can calculate that the eigenstates $|+\rangle$ and $|-\rangle$ are:

$|+\rangle = \frac{1}{\sqrt{2}}(e^{-i\varphi}|g\rangle + |e\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|g\rangle - e^{i\varphi}|e\rangle)$, and that $|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|g\rangle + i|e\rangle)$ if $\varphi = 0$, so in an equal superposition of $|g\rangle$ and $|e\rangle$.

EXERCISE 2

1

- $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

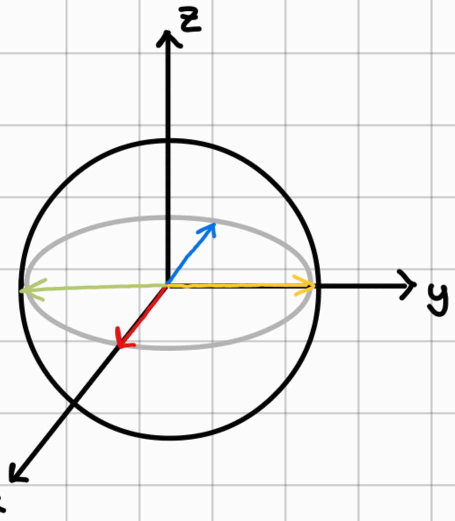
$\sin\left(\frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)$

$\Rightarrow \frac{\theta}{2} = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{2}$

$\varphi = 0$

$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\theta = \frac{\pi}{2}$, $\varphi = \pi$



$\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \Rightarrow \theta = \frac{\pi}{2}, \varphi = \frac{\pi}{2}$

$\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \Rightarrow \theta = \frac{\pi}{2}, \varphi = -\frac{\pi}{2}$

2 $H' = \frac{\hbar}{2}\begin{pmatrix} -\Delta & \Omega e^{-i\varphi} \\ \Omega e^{i\varphi} & \Delta \end{pmatrix}$, $U = e^{-iH't/\hbar}$

- $\underline{\Delta = 0, \varphi = 0}$

$H' = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} = \frac{\hbar}{2}\Omega\sigma_x \Rightarrow U = e^{-i\Omega t\sigma_x/2}$

= rotation around x axis by angle Ωt

- $\underline{\Delta = 0, \phi = \pi/2}$

$H' = \frac{\hbar}{2}\begin{pmatrix} 0 & -i\Omega \\ i\Omega & 0 \end{pmatrix} = \frac{\hbar}{2}\Omega\sigma_y \Rightarrow U = e^{-i\Omega t\sigma_y/2}$ rotation around y axis by Ωt

- $\underline{\Delta \neq 0, \Omega = 0}$

$H' = \frac{\hbar}{2}\begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix} = -\frac{\hbar}{2}\Delta\sigma_z \Rightarrow U = e^{i\Delta t\sigma_z/2}$ rotation around z axis by $-\Delta t$

3 When we drive exactly on resonance, we can choose to around an arbitrary axis on the xy plane by controlling the phase. In addition, by introducing a finite detuning $\Delta \neq 0$ we can also rotate around the z axis. This creates a basis set of two rotations that allow any arbitrary rotation in the Bloch sphere.