Introduction to quantum science and technology: qubit platforms
Session 1

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Two-level systems

As we know from quantum mechanics, quantum systems often have discrete energy levels. This is true for instance for electrons in neutral atoms and ions, for spins in a magnetic field, for photons confined in superconducting circuits, and so on. If the number of discrete levels is only two (as for a spin-1/2 particle in a magnetic field), or, for multi-level systems, if we only restrict our interest to two energy levels, we obtain a so-called *two-level system*, which is the building block of qubits. Using the Dirac notation, we can denote these two states as $|g\rangle$ and $|e\rangle$ (for "ground" and "excited"), or as $|0\rangle$ and $|1\rangle$, or also as $|{\downarrow}\rangle$ and $|{\uparrow}\rangle$. These are conventions that you will typically find in the literature and we can use them interchangeably.

Working with a quantum system described by two energy levels effectively means working in a 2-dimensional space, which implies that quantum states will be represented by 2-dimensional vectors, while quantum operators by 2×2 matrices. The set $\{|g\rangle, |e\rangle\}$ (or equivalently $\{|0\rangle, |1\rangle\}$ or $\{|{\downarrow}\rangle, |{\uparrow}\rangle\}$) make a basis for this 2-dimensional space. Using the vector notation in this basis, we can write these basis states as:

$$|g\rangle \equiv |0\rangle \equiv |{\downarrow}\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |e\rangle \equiv |1\rangle \equiv |{\uparrow}\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{1}$$

For the next exercise, we denote the basis as $\{|g\rangle, |e\rangle\}$, and we assume that $|g\rangle$ denotes an energy level with energy E_g, $|e\rangle$ an energy level of energy E_e and that $E_g < E_e$, as is depicted in figure 1. Furthermore, let us define $E_e - E_g \equiv \hbar\omega_0$ and set the reference for zero energy such that $E_g = -\hbar\omega_0/2$ and $E_e = \hbar\omega_0/2$. ω_0 is usually called *qubit frequency*. With these choices, the Hamiltonian of the two-level system is:

$$H_0 = \frac{\hbar\omega_0}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right) = \begin{pmatrix}-\hbar\omega_0/2 & 0\\ 0 & \hbar\omega_0/2\end{pmatrix}. \tag{2}$$

The most generic state $|\psi\rangle$ of a two-level system can be written as a linear superposition of $|g\rangle$ and $|e\rangle$:

$$|\psi\rangle = \alpha|g\rangle + \beta|e\rangle. \tag{3}$$

If ψ is *normalized*, then $\langle\psi|\psi\rangle = 1$, hence $|\alpha|^2 + |\beta|^2 = 1$.

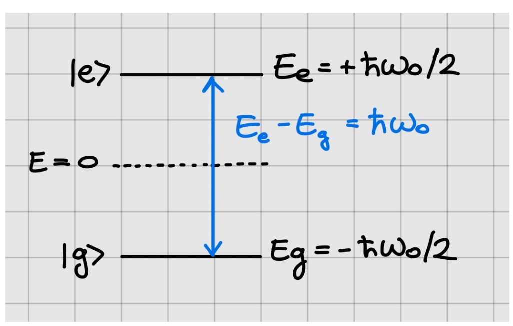

Figure 1: Energy diagram of a two-level system.

It is very useful to define the so-called *Pauli matrices*:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |e\rangle\langle g| + |g\rangle\langle e|; \tag{4}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\,|e\rangle\langle g| - i\,|g\rangle\langle e|; \tag{5}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |g\rangle\langle g| - |e\rangle\langle e|. \tag{6}$$

Exercise 1: Manipulating quantum states: driven two-level system

Manipulating a qubit means being able to change the quantum state in which the system is found. This is often done, in practice, by coupling the two-level system with an oscillating electromagnetic field, also called *drive*, oscillating at a frequency ω_d. The Hamiltonian describing the interaction between the two-level system and the drive is:

$$H_\mathrm{d}(t) = \hbar\Omega\cos(\omega_\mathrm{d}t - \phi)(|g\rangle\langle e| + |e\rangle\langle g|), \tag{7}$$

where Ω, called *Rabi frequency*, represents the strength of the interaction and its analytical expression depends on the physical nature of the two-level system and how it couples to the electromagnetic field. ϕ is a phase in the drive oscillations. The total Hamiltonian of the system is:

$$H = H_0 + H_\mathrm{d} = \frac{\hbar\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \hbar\Omega\cos(\omega_\mathrm{d}t - \phi)(|g\rangle\langle e| + |e\rangle\langle g|). \tag{8}$$

1. Rewrite Hamiltonian 8 in matrix form on the basis $\{|g\rangle, |e\rangle\}$ and then in terms of the Pauli operators . Try to explain intuitively how the interaction Hamiltonian acts on the system.

We are interested in studying the time evolution of the system under this Hamiltonian. However, since the Hamiltonian is time-dependent, this task is not so straightforward. The problem is simplified, though, if one adopts a reference frame *rotating at the drive frequency* (the exact meaning of "rotating" will become more clear later when talking about the Bloch sphere). Practically speaking, the change of reference frame is done by defining the unitary transformation:

$$U = e^{-i\omega_\mathrm{d}t\sigma_z/2} = e^{-i\frac{\omega_\mathrm{d}t}{2}(|g\rangle\langle g| - |e\rangle\langle e|)} \tag{9}$$

and by transforming H according to:

$$H \to H' = UHU^\dagger + i\hbar\frac{\partial U}{\partial t}U^\dagger. \tag{10}$$

2. (Optional) Perform the transformation defined in eq. 10 and show that H' can be expressed as:

$$H' = \frac{\hbar(\omega_0 - \omega_\mathrm{d})}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{\hbar\Omega}{2}\left[\left(e^{-i\phi} + e^{-2i\omega_\mathrm{d}t + i\phi}\right)|g\rangle\langle e| + \left(e^{i\phi} + e^{2i\omega_\mathrm{d}t - i\phi}\right)|e\rangle\langle g|\right] \tag{11}$$

From now on, we can define $\Delta = \omega_0 - \omega_\mathrm{d}$, the qubit-to-drive detuning. The first term in H' contains the diagonal terms, while the second term starting with $\hbar\Omega/2$ contains the off-diagonal terms. We can see that the off-diagonal terms contain a time-independent part with $e^{\pm i\phi}$, and a time-dependent part evolving at twice the drive frequency ($e^{\pm(2i\omega_\mathrm{d}t-i\phi)}$). At this point it is common to do the so-called *rotating-wave approximation* (RWA), which consists of neglecting the terms evolving at twice the drive frequency. Intuitively, on the time scales of $1/\omega_\mathrm{d}$, terms evolving as $2\omega_\mathrm{d}t$ average out to zero because they are twice as fast.

3. Rewrite H' by applying the rotating wave approximation, both in the bra-ket notation and in matrix form. In matrix form, you should find:
$$H' = \frac{\hbar}{2}\begin{pmatrix} -\Delta & \Omega e^{-i\phi} \\ \Omega e^{i\phi} & \Delta \end{pmatrix} \tag{12}$$

4. Diagonalize H' to find its eigen-energies $E_\pm$ and its eigen-vectors $|+\rangle, |-\rangle$ (hint: the eigenvectors can be written in multiple ways, but some ways are a more convenient choice for the next question).

5. Starting from the initial state $|\psi_0\rangle = |g\rangle$, study the time evolution $|\psi(t)\rangle$ of a driven two-level system. Show that the probability $P_e(t) = |\langle e|\psi(t)\rangle|^2$ of finding the system in the excited state at time t is:
$$P_e(t) = \frac{\Omega^2}{\Delta^2+\Omega^2}\sin^2\left(\frac{\sqrt{\Delta^2+\Omega^2}}{2}t\right) \tag{13}$$
which describes so-called *Rabi oscillations*. (Hint: express $|g\rangle$ in terms of $|+\rangle$ and $|-\rangle$ and then apply the time evolution operator in the rotating frame $U_\mathrm{ev}(t) = e^{-iH't/\hbar}$.)

6. Plot qualitatively $P_e(t)$ for the three cases $\Delta = 0$, $\Delta = \Omega$, $\Delta = 3\Omega$.

7. A so-called π-pulse consist in a drive at $\Delta = 0$ sent for a duration t such that $\Omega t = \pi$. Explain what happens to the two-level system under this drive, if $|\psi(t=0)\rangle = |g\rangle$. And what happens instead if a $\pi/2$-pulse is sent, i.e. such that $\Omega t = \pi/2$?

Exercise 2: The Bloch sphere

An arbitrary state of a two-level system can be written as:
$$|\psi(t)\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle. \tag{14}$$
with $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi]$. This representation is useful because we can represent the state as a vector pointing to the surface of a sphere with unitary radius, using ϕ and θ as conventional polar coordinates as shown in figure 2. In this representation, the two states $|0\rangle$ and $|1\rangle$ are at the poles of the sphere. The Bloch sphere is a useful way to easily visualize the state of the system and how it evolves with time, as the θ angle represents "how much" $|g\rangle$ and $|e\rangle$ are in quantum superposition and the angle ϕ represents the quantum phase between them in the quantum state. All points on the surface of the Bloch sphere then represent all the pure states that the qubit can take.

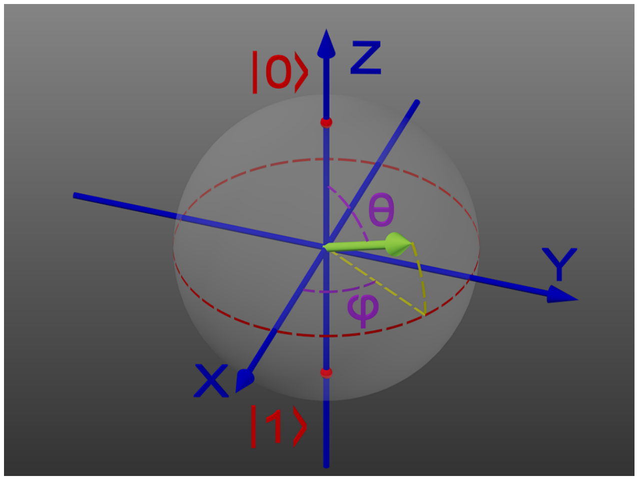

Figure 2: Graphical representation of the Bloch sphere.

1. Draw the following states on the Bloch sphere:
   - $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
   - $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
   - $\frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$
   - $\frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$

2. An important concept is that the following operators rotate states on the Bloch sphere by an angle Θ respectively around the x, y and z axis:

$$R_x(\Theta) = e^{-i\Theta\sigma_x/2}, \tag{15}$$
$$R_y(\Theta) = e^{-i\Theta\sigma_y/2}, \tag{16}$$
$$R_z(\Theta) = e^{-i\Theta\sigma_z/2}. \tag{17}$$

   Now consider a driven two-level system and recall that its Hamiltonian, after going into the rotating frame and applying the RWA, is the one written in eq. 12. According to quantum mechanics, the time-evolution of the system is obtained by applying the operator $U = e^{-iH't/\hbar}$. Describe what this operator does to a general state on the Bloch sphere in the following cases:
   - $\Delta = 0$, $\phi = 0$
   - $\Delta = 0$, $\phi = \pi/2$
   - $\Delta \neq 0$, $\Omega = 0$

   (Hint: rewrite H' in terms of the Pauli operators and try to identify one of the rotation operators in U.)

3. Explain how one can perform an arbitrary gate on a qubit, i.e. an arbitrary rotation in the Bloch sphere.