Introduction to quantum science and technology: qubit platforms
Session 2

Assistants : elena.acinapura@epfl.ch

Superconducting circuits

Introduction

In the lecture you have seen that superconducting circuits are a platform where it's possible to confine electromagnetic fields and make them have discrete energy levels. You have seen that if one takes an LC circuit as the one depicted in figure 1, its Hamiltonian is:

$$H = \hbar\omega_0 a^\dagger a, \tag{1}$$

where $\omega_0 = 1/\sqrt{LC}$ is typically designed to be in the few GHz range. This Hamiltonian has exactly the same form of the Hamiltonian of an harmonic oscillator, which is not surprising because the electric and magnetic field in and LC circuit oscillate periodically. The operators a and $a^\dagger$ are ladder operators for these oscillating fields, and the excitations of the electromagnetic field, given by $a^\dagger a$, are called photons.

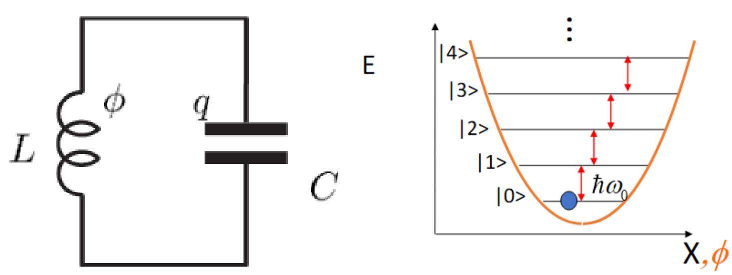

Figure 1: LC circuit as an harmonic oscillator.

As you well know from your quantum mechanics course, a quantum harmonic oscillator described by Hamiltonian 1 has energy levels that are equally spaced by $\hbar\omega_0$, as depicted in figure 1. This prevents one from being able to operate a perfect harmonic oscillator as a two-level system, because if you want to couple two levels with a drive at a frequency ω_0, this drive would also drive other transitions, since the energy levels are all equally spaced. For this reason, people came up with the idea of making the LC circuit *nonlinear* by replacing the inductor with a *Josephson junction*, depicted in figure 2. The current $I(t)$ through a Josephson junction and the voltage $V(t)$ across it must satisfy the so-called Josephson relations:

$$\begin{cases} I(t) = I_c \sin(\delta(t)), \\ V(t) = \dfrac{\Phi_0}{2\pi}\dot{\delta}(t), \end{cases} \tag{2}$$

where δ is the phase difference between the two superconducting electrodes of the junction, $\Phi_0 = h/(2e)$ is the superconducting flux quantum and I_c is the critical current of the junction, which is

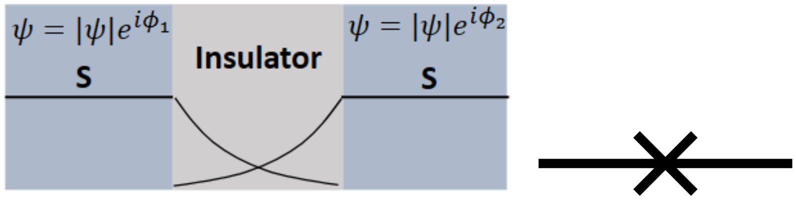

Figure 2: A Josephson junction and its circuit symbol.

a characteristic parameter of the junction which depends on its design parameters. It is common to define a *flux variable* $\Phi(t)$ such that:

$$\delta(t) = 2\pi \frac{\Phi(t)}{\Phi_0} \tag{3}$$

so that the Josephson relations can also be rewritten as:

$$\begin{cases} I(t) = I_c \sin\left(2\pi \frac{\Phi(t)}{\Phi_0}\right), \\ V(t) = \dot{\Phi}(t). \end{cases} \tag{4}$$

Exercise 1: The Cooper Pair Box Hamiltonian

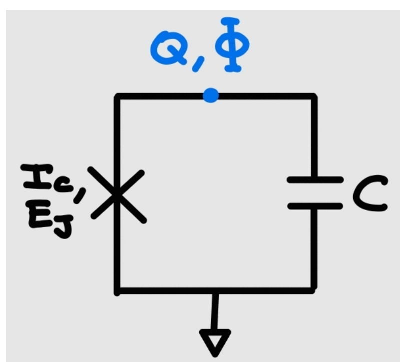

Figure 3: Schematic of a Cooper Pair Box.

Consider the circuit described in figure 3, representing an LC circuit where the inductor has been replaced with a Josephson junction.

1. Taking inspiration from the calculations that you have seen in the lecture for the LC circuit, show that the Lagrangian of the system expressed using the flux Φ is:

$$\mathcal{L} = \frac{C\dot{\Phi}^2}{2} + E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right) \tag{5}$$

   where $E_J = I_C\Phi_0/(2\pi)$.

2. Do the Legendre transformation to find the charge Q as the conjugate variable of Φ:

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \tag{6}$$

and then show that the Hamiltonian of the system is:

$$H = Q\dot{\Phi} - \mathcal{L} = \frac{Q^2}{2C} - E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right). \tag{7}$$

3. Since we are dealing with a superconducting circuit where the charge carriers are Cooper pairs (i.e. pairs of two electrons), Q must be an even multiple of the elementary charge e. Therefore we can introduce a number of Cooper pairs n such that $Q = 2en$. Moreover, we can define the *charging energy* as $E_c = e^2/(2C)$, i.e. the energy required to increase Q by one electron. Since Q and Φ are conjugate variables, we would like to keep writing the Hamiltonian using n and its conjugate variable. One can verify that the conjugate variable of n is δ. Rewrite H in terms of n, δ, E_J and E_C.

Now we extend a bit the circuit to represent a more realistic scenario, where there may be some voltage fluctuations on the ungrounded part of the circuit. We can depict this situation as in figure 4, where a gate voltage is applied capacitively to the Cooper pair box island.

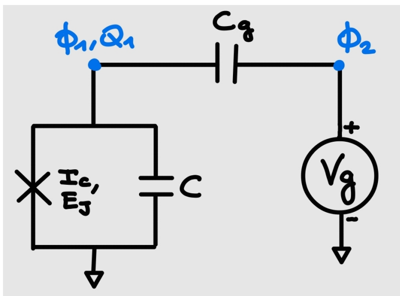

Figure 4: Schematic of a CPB with an applied gate voltage.

4. Find the Lagrangian of the system in this new configuration, then follow the same steps as above to show that the Hamiltonian of the system is:

$$H = \frac{(\hat{Q}_1 + C_g V_g)^2}{2(C + C_g)} - E_J \cos\left(2\pi \frac{\hat{\Phi}_1}{\Phi_0}\right) - \frac{C_g V_g^2}{2}, \tag{8}$$

We can see that the last term in Hamiltonian 8 is just a constant energy offset to the Hamiltonian, which does not determine the physical behavior of the circuit, therefore we can neglect it from now on. It is convenient to define $Q_g = -C_g V_g$, $C_\Sigma = C + C_g$ and then express the charges in terms of the Cooper pair numbers: $\hat{Q}_1 = 2e\hat{N}$, $Q_g = 2en_g$. We can finally rewrite the Hamiltonian as:

$$\hat{H} = 4E_C(\hat{N} - n_g)^2 - E_J \cos\left(\hat{\delta}\right) + const. \tag{9}$$

where now $E_C = e^2/(2C_\Sigma)$.

Exercise 2: The CPB spectrum and the transmon limit

Let us now investigate the spectrum of the Cooper pair box Hamiltonian:

$$\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J \cos\left(\hat{\delta}\right) \tag{10}$$

where $\hat{N}$ is the charge operator, $\hat{\delta}$ the phase operator, E_C the charging energy, E_J the Josephson energy and n_g an offset charge. $\hat{N}$ and $\hat{\delta}$ are conjugate operators, meaning that their commutator is $[\hat{\delta}, \hat{N}] = i$.

1. (Optional) Show that $[\hat{N}, e^{i\hat{\delta}}] = e^{i\hat{\delta}}$.
   Hint: Use a power series to express the exponential.
2. (Optional) Show that $e^{i\hat{\delta}} |m\rangle = |m+1\rangle$, where $|m\rangle$ is an eigenstate of the charge operator, i.e. $\hat{N}|m\rangle = m|m\rangle$.
   Hint: Make use of the identity shown in 1.
3. Use the identities from 1. and 2. to show that the CPB Hamiltonian in the charge basis reads

$$\hat{H}_{CPB} = \sum_{m=-\infty}^{+\infty} \left[4E_C(m - n_g)^2 |m\rangle\langle m| - \frac{1}{2}E_J(|m+1\rangle\langle m| + |m\rangle\langle m+1|)\right] \tag{11}$$

   Explain intuitively what the different terms do in the Hamiltonian.
4. Figure 5 shows the energy levels of the CPB Hamiltonian with respect to the bias n_g for different regimes of $\frac{E_J}{E_C} = 1.0, 5.0, 10.0, 50.0$. You can obtain these plots by numerically diagonalizing the Hamiltonian in the charge basis using python packages like Qutip. Explain which regime is the most convenient choice to operate two energy levels as a qubit.

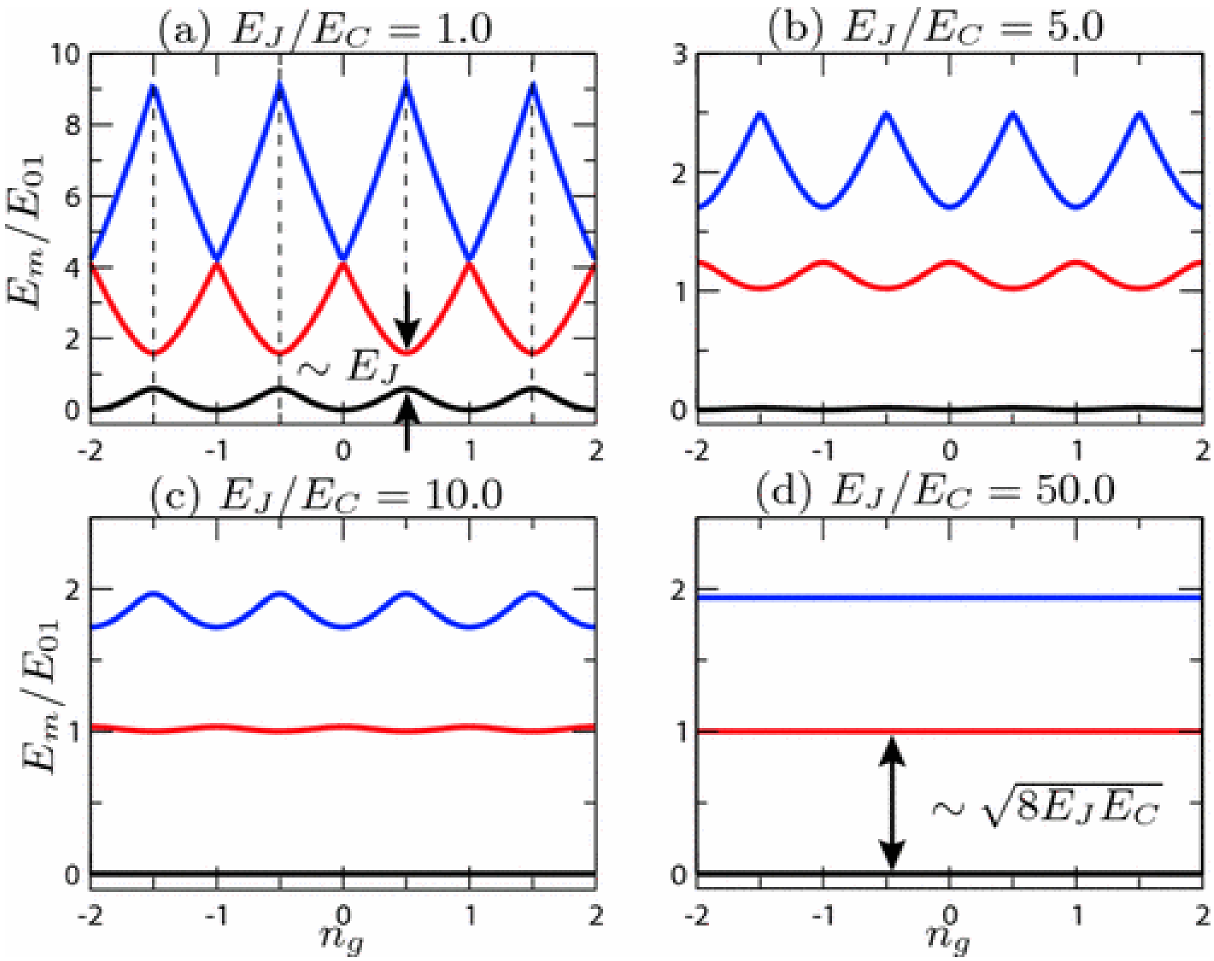

Figure 5: Plot of the energy levels of a CPB with bias voltage n_g for different regimes.

Exercise 3: Identifying components on a superconducting chip

Let us finally have a look a real superconducting chip in order to get familiar with the different components and the role they play in measurements. Figures 6 and 7 show a microscope image of the famous 17-superconducting-qubit chip taken from the paper "Realizing repeated quantum error correction in a distance-three surface code" by Sebastian Krinner et al.

1. Can you identify each component on the chip and what role they play in a measurement? Each component to identify is in a different color.

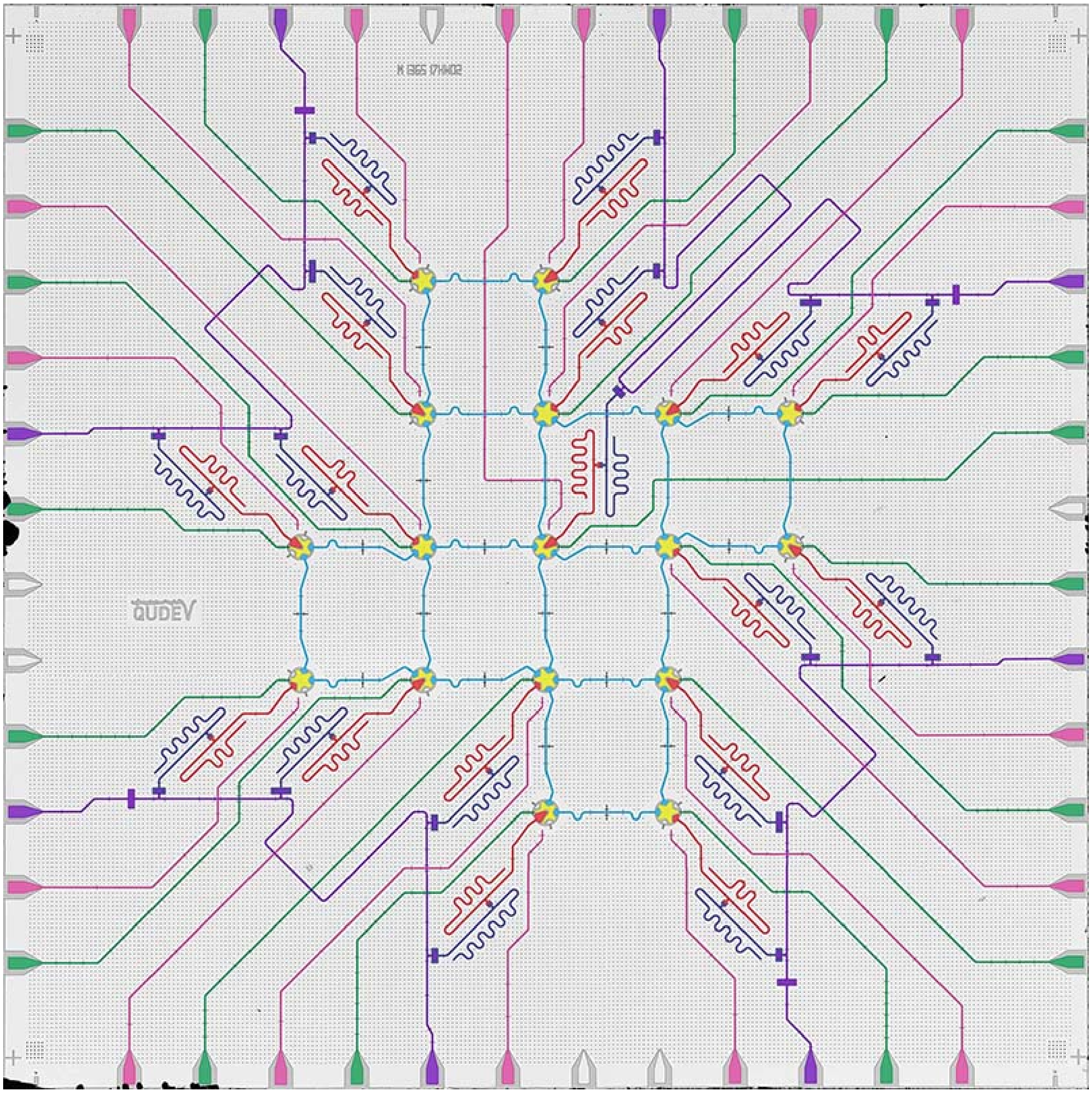

Figure 6: Picture of a superconducting 17 qubit chip.

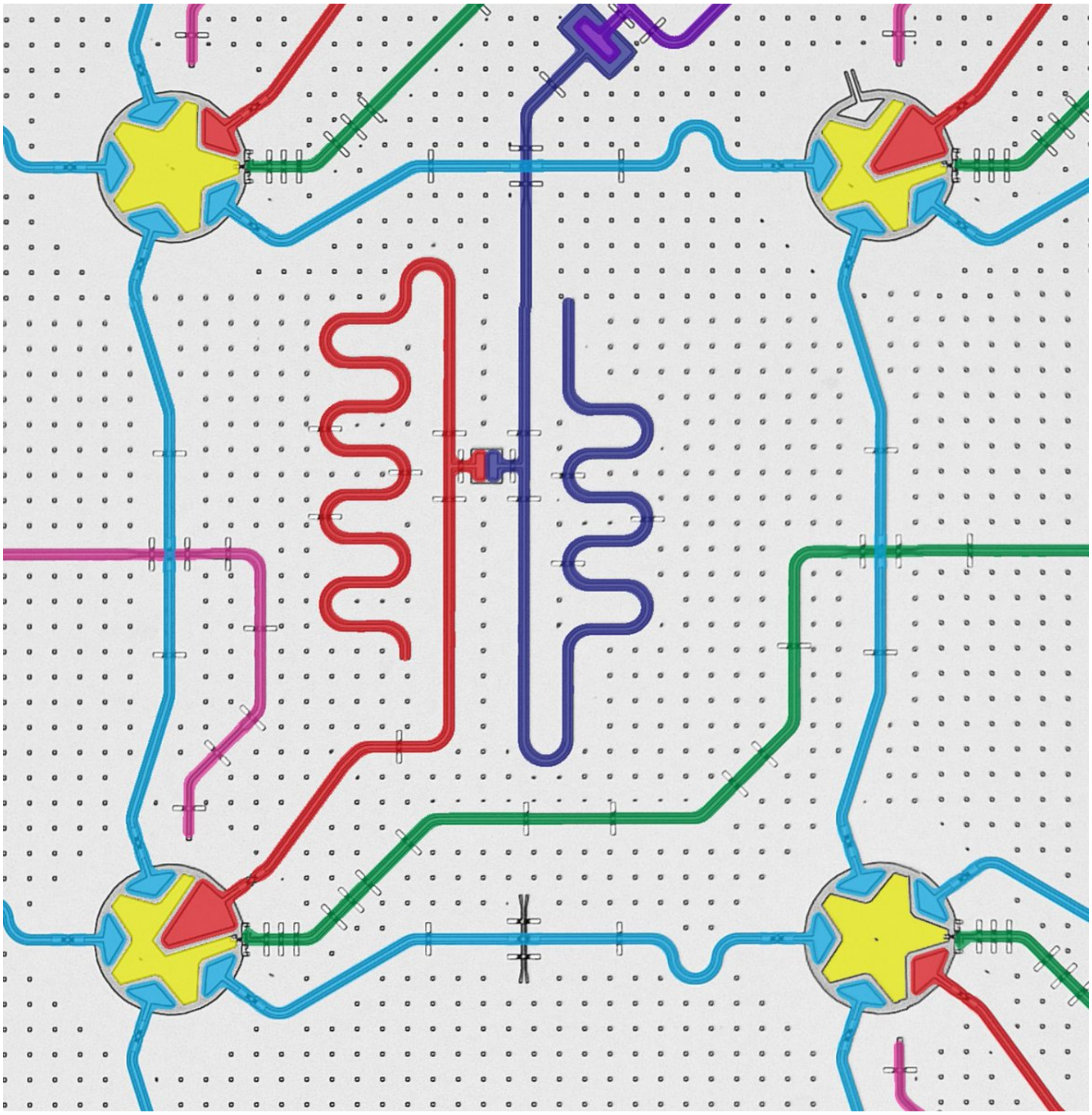

Figure 7: Picture of a superconducting 17 qubit chip (zoomed in).