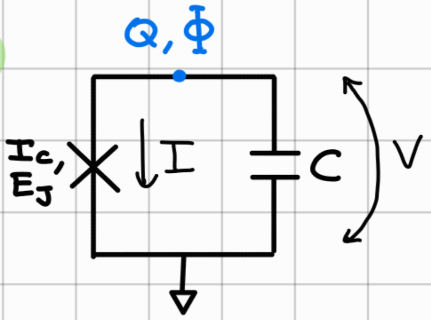
EXERCISE 1

1



From Lagrangian mechanics we know:

$$\mathcal{L} = \underset{\text{kinetic energy}}{T} - \underset{\text{potential energy}}{U}$$

We can choose to express this in terms of Q or ϕ. If we choose to use ϕ, then one usually identifies the "kinetic" energy term with the energy stored in the capacitor, and the "potential" energy term with the energy in the Josephson junction. Rembering the Josephson relations:

$$\begin{cases} I = I_c \sin\left(2\pi \frac{\Phi}{\Phi_0}\right) & (1) \\ V = \dot{\phi} & (2) \end{cases}$$

We have:

$$T = \frac{1}{2} C V^2 \overset{(1)}{=} \frac{1}{2} C \dot{\phi}^2$$

and:

$$U = \int I(t) V(t)\, dt \overset{(1)(2)}{=} \int I_c \sin\left(2\pi \frac{\phi(t)}{\Phi_0}\right) \dot{\phi}(t)\, dt \overset{\text{change variable} \to \phi}{=}$$

$$= \int I_c \sin\left(2\pi \frac{\phi}{\Phi_0}\right) d\phi = -\underbrace{\frac{\Phi_0 I_c}{2\pi}}_{E_J} \cos\left(2\pi \frac{\phi}{\Phi_0}\right)$$

So

$$\mathcal{L} = \frac{C\dot{\phi}^2}{2} + E_J \cos\left(2\pi \frac{\phi}{\Phi_0}\right)$$

2 $Q = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = C\dot{\phi} \Rightarrow \dot{\phi} = \frac{Q}{C}$

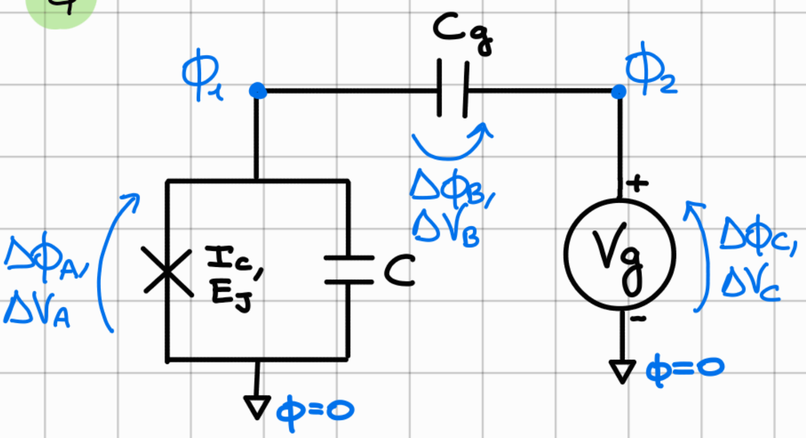
$$\Rightarrow H = Q\dot{\phi} - \mathcal{L} = Q\dot{\phi} - \frac{C\dot{\phi}^2}{2} - E_J \cos\left(2\pi \frac{\phi}{\Phi_0}\right) =$$

$$= \frac{Q^2}{C} - \frac{Q^2}{2C} - E_J \cos\left(2\pi \frac{\phi}{\phi_0}\right) = \frac{Q^2}{2C} - E_J \cos\left(2\pi \frac{\phi}{\phi_0}\right)$$

3 $Q = 2en$, $\phi = \frac{\phi_0}{2\pi}\delta$

$$\Rightarrow H = \frac{4e^2 n^2}{\underbrace{2C}_{E_c}} - E_J \cos\left(2\pi \frac{\phi}{\phi_0}\right) = 4E_c n^2 - E_J \cos(\delta)$$

4



We define the flux differences:

$$\Delta\phi_A = \phi_1 - 0 = \phi_1$$

$$\Delta\phi_B = \phi_2 - \phi_1$$

$$\Delta\phi_C = \phi_2 - 0 = \phi_2$$

The relation between the voltages and the fluxes is:

$$\Delta V_A = \Delta\dot{\phi}_A = \dot{\phi}_1$$

$$V_g = \Delta V_c = \Delta\dot{\phi}_c = \dot{\phi}_2$$

$$\Delta V_B = \Delta\dot{\phi}_B = \dot{\phi}_2 - \dot{\phi}_1 = V_g - \dot{\phi}_1$$

Let's write the Lagrangian:

$$\mathcal{L} = \frac{C_g \Delta V_b^2}{2} + \frac{C \Delta V_a^2}{2} + E_J \cos\left(2\pi \frac{\Delta\Phi_A}{\Phi_0}\right)$$

$$= \frac{C_g (\Delta\dot{\Phi}_B)^2}{2} + \frac{C(\Delta\dot{\Phi}_A)^2}{2} + E_J \cos\left(2\pi \frac{\Phi_1}{\Phi_0}\right)$$

$$= \frac{C_g}{2}(\dot{\Phi}_2 - \dot{\Phi}_1)^2 + \frac{C}{2}\dot{\Phi}_1^2 + E_g \cos\left(2\pi \frac{\Phi_1}{\Phi_0}\right)$$

$$= \frac{C_g}{2}(V_g - \dot{\Phi}_1)^2 + \frac{C}{2}\dot{\Phi}_1^2 + E_g \cos\left(2\pi \frac{\Phi_1}{\Phi_0}\right)$$

Legendre transformation:

$$Q_1 = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_1} = C\dot{\Phi}_1 - C_g(V_g - \dot{\Phi}_1) = (C + C_g)\dot{\Phi}_1 - C_g V_g$$

$$\Rightarrow \quad \dot{\Phi}_1 = \frac{Q + C_g V_g}{C + C_g}$$

Hamiltonian:

$$\mathcal{H} = Q_1 \dot{\Phi}_1 - \mathcal{L} =$$

$$= C\dot{\Phi}_1^2 + C_g\dot{\Phi}_1^2 - \cancel{C_g V_g \dot{\Phi}_1} - \frac{C_g}{2}V_g^2 - \frac{C_g}{2}\dot{\Phi}_1^2 + \cancel{C_g V_g \dot{\Phi}_1} - \frac{C}{2}\dot{\Phi}_1^2 +$$

$$- E_J \cos\left(2\pi \frac{\Phi_1}{\Phi_0}\right) =$$

$$= \frac{C}{2}\dot{\Phi}_1 + \frac{C_g}{2}\dot{\Phi}_1^2 - \frac{C_g V_g^2}{2} - E_J \cos\left(2\pi \frac{\Phi_1}{\Phi_0}\right) =$$

$$= \frac{1}{2}(C + C_g)\dot{\Phi}_1^2 - \frac{C_g V_g^2}{2} - E_J \cos\left(2\pi \frac{\Phi_1}{\Phi_0}\right) =$$

$$= \frac{(Q_1 + C_g V_g)^2}{2(C + C_g)} - E_J \cos\left(2\pi \frac{\Phi_1}{\Phi_0}\right) - \frac{C_g V_g^2}{2}$$

$\hat{Q}_1 = 2e\hat{N}$, $Q_g = -C_g V_g = 2e n_g$

$$\Rightarrow \quad H = 4E_C(\hat{N}-n_g)^2 - E_J \cos(\hat{\delta})$$

EXERCISE 2

1 $$[N, e^{i\delta}] = [N, \sum_{n=0}^{\infty} \frac{1}{n!}(i\delta)^n] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \underline{\underline{[N,\delta^n]}} =$$

$$= \sum_{n=1}^{+\infty} \frac{i^n}{n!}[N,\delta^n] \quad \text{since } [N,\delta^0] = [N,\mathbb{1}] = 0$$

$$\underline{\underline{[N,\delta^n]}} = [N, \delta\delta^{n-1}] = [N,\delta]\delta^{n-1} + \delta[N,\delta^{n-1}] =$$

$$= -i\delta^{n-1} + \delta[N,\delta^{n-1}] =$$

$$= -i\delta^{n-1} + \delta(-i\delta^{n-2} + \delta^{n-1}[N,\delta^{n-2}]) = \dots$$

$$= -ni\delta^{n-1}$$

$$\Rightarrow [N, e^{i\delta}] = \sum_{n=1}^{+\infty} \frac{i^n}{n!} n(-i)\delta^{n-1} = \sum_{n=1}^{+\infty} \frac{i^{n-1}}{(n-1)!}\delta^{n-1} =$$

$$= \sum_{n=0}^{+\infty} \frac{i^n}{n!}\delta^n = e^{i\delta}$$

2 Let's call $|\psi\rangle = e^{i\delta}|m\rangle$. Let's see the result of $\hat{N}|\psi\rangle$:

$$N|\psi\rangle = Ne^{i\delta}|m\rangle = (e^{i\delta}N + [N, e^{i\delta}])|m\rangle =$$

$$= e^{i\delta}N|m\rangle + e^{i\delta}|m\rangle = e^{i\delta}m|m\rangle + e^{i\delta}|m\rangle =$$

$$= (m+1)e^{i\delta}|m\rangle = (m+1)|\psi\rangle$$

But $N|\psi\rangle = (m+1)|\psi\rangle \Rightarrow |\psi\rangle = |m+1\rangle$!

3. We can exploit the identity:

$$\mathbb{1} = \sum_{m=-\infty}^{+\infty} |m\rangle\langle m|$$

to rewrite:

$$H = \mathbb{1}\, H\, \mathbb{1} = \left(\sum_{m'=-\infty}^{+\infty} |m'\rangle\langle m'|\right) H \left(\sum_{m=-\infty}^{+\infty} |m\rangle\langle m|\right) =$$

$$= \sum_{m',m} \Big(|m'\rangle\langle m'| \, 4E_c(\hat{N}-n_g)^2 |m\rangle\langle m| \, +$$

$$- |m'\rangle\langle m'| \, E_J \cos(\delta) |m\rangle\langle m| \Big) =$$

$$= \sum_{m',m} \Big(4E_c(m-n_g)^2 |m'\rangle \underbrace{\langle m'|m\rangle}_{\substack{\neq 0 \text{ only} \\ \text{if } m'=m \\ \text{bc. orthogonal}}} \langle m| - \frac{E_J}{2} |m'\rangle\langle m'| (e^{i\delta} + e^{-i\delta}) |m\rangle\langle m| \Big) =$$

$$= \sum_m 4E_c(m-n_g)^2 |m\rangle\langle m| - \sum_{m,m'} \frac{E_J}{2} \Big(|m'\rangle \overbrace{\langle m'+1|m\rangle}^{\neq 0 \text{ if } m'+1=m} \langle m| +$$

$$+ |m'\rangle \underbrace{\langle m'|m+1\rangle}_{\neq 0 \text{ if } m'=m+1} \langle m| \Big) =$$

$$= \sum_m \Big[\underbrace{4E_c(m-n_g)^2 |m\rangle\langle m|}_{\text{Diagonal terms in the } |m\rangle \text{ basis}} - \underbrace{\frac{E_J}{2} \big(|m\rangle\langle m+1| + |m+1\rangle\langle m| \big)}_{\text{"Hopping terms"}} \Big]$$

Diagonal terms in the $|m\rangle$ basis

"Hopping terms", coupling $|m\rangle$ with $|m+1\rangle$. This term describes the tunneling of 1 Cooper pair in/out of the CPB island through the Josephson junction

4 The most convenient choice is $E_J / E_C \gg 1$, so that the energies do not depend to n_g. This is because n_g can fluctuate due to charge noise on the qubit island.