# Phase estimation algorithm

Let $U$ be a $2^n \times 2^n$ unitary transformation

Let also $|u\rangle$ be an eigenvector of $U$

with corresponding eigenvalue $\lambda$: $U|u\rangle = \lambda|u\rangle$

As $U$ is unitary, $U^\dagger U = I$, so

$$1 = \langle u|u\rangle = \langle u|U^\dagger U|u\rangle = \| U|u\rangle \|^2 = |\lambda|^2 \cdot \underbrace{\| |u\rangle \|^2}_{=1}$$

and therefore $|\lambda| = 1$: $\lambda = e^{2\pi i \varphi}$, $0 \leq \varphi \leq 1$

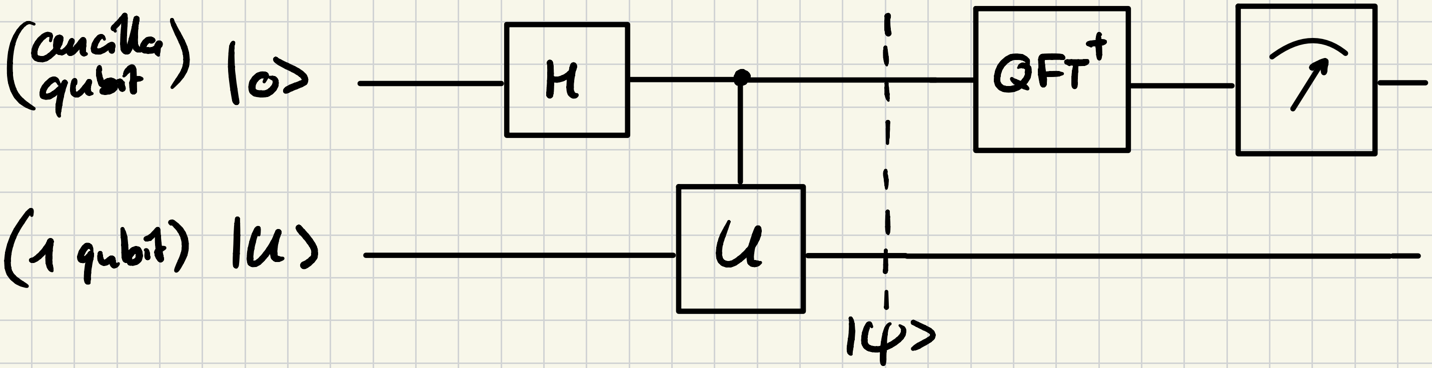
Our aim is to build a quantum circuit allowing to estimate the phase $\varphi$.

Let us assume here for simplification that $\varphi = \varphi_1 \cdot 2^{-1} + \varphi_2 \cdot 2^{-2} + \dots + \varphi_n 2^{-n}$ (binary exp., $\varphi_i \in \{0,1\}$)

(<u>ex</u>: $\varphi = 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = 0.375$)

and for further simplification, let us first assume that $n=1$ (so $\varphi = \varphi_1 \cdot 2^{-1}$ with $\varphi_1 \in \{0,1\}$)

Here is the phase estimation circuit (for $n=1$):




where recall that $|u\rangle \in \mathbb{C}^2$ is the eigenvector of $U$

and $\quad U|u\rangle = \exp(2\pi i \varphi)|u\rangle$

with $\quad \varphi = \varphi_1 \cdot 2^{-1}$ and $\varphi_1 \in \{0,1\}$

(so either $\varphi = 0$ or $\varphi = 0.5$)

Let us compute the intermediary state $|\psi\rangle$ :

$$|\psi\rangle = \text{Control-}U \left( H|0\rangle \otimes |u\rangle \right)$$

$$= \text{Control-}U \left( \frac{1}{\sqrt{2}} |0\rangle \otimes |u\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes u\rangle \right)$$

$$= \frac{1}{\sqrt{2}} |0\rangle \otimes |u\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes \underbrace{U|u\rangle}_{= \lambda |u\rangle}$$

$$= \frac{1}{\sqrt{2}} |0\rangle \otimes |u\rangle + \frac{\exp(2\pi i \varphi)}{\sqrt{2}} |1\rangle \otimes |u\rangle$$

$$= \frac{1}{\sqrt{2}} \left( |0\rangle + \exp(2\pi i \varphi) |1\rangle \right) \otimes |u\rangle$$

Recalling that $\varphi = \frac{\varphi_1}{2}$, we obtain

$$|\psi\rangle = \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle + \exp\left(\frac{2\pi i \varphi_1}{2}\right)|1\rangle\right)}_{= \text{expression of the QFT for } n=1\,!} \otimes |u\rangle$$

so $|\psi\rangle = QFT\,|\varphi_1\rangle \otimes |u\rangle$, and the output of the circuit is

$$QFT^{+}\left(QFT\,|\varphi_1\rangle\right) \otimes |u\rangle = |\varphi_1\rangle \otimes |u\rangle$$

and the measurement of the first qubit gives exactly $\varphi_1$ in this case.

Let us now consider the general case $n \geq 1$:

In this case, the action of $U$ on $|u\rangle$ is

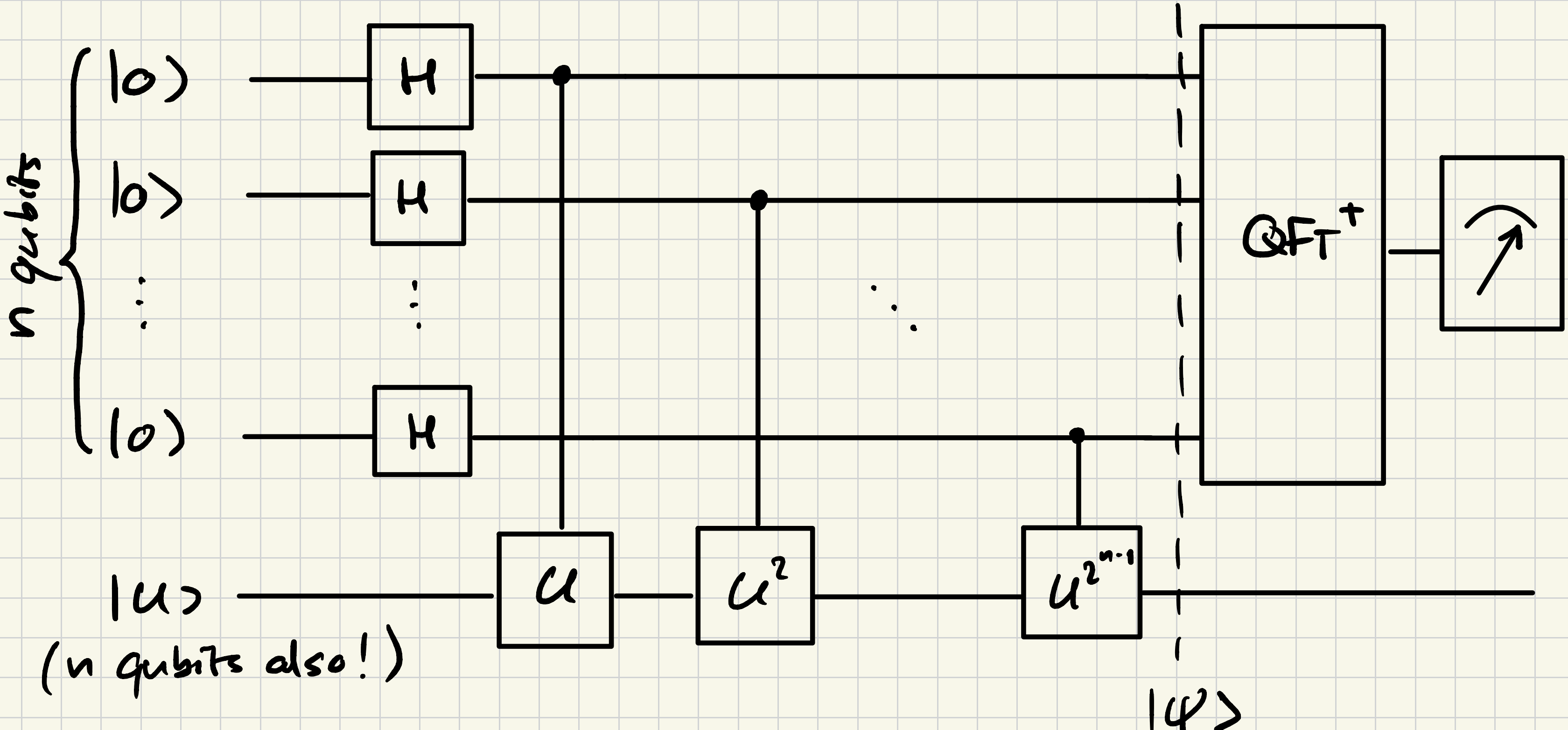
$$U\,|u\rangle = \exp(2\pi i \varphi)\,|u\rangle \quad \text{where } \varphi = \sum_{j=1}^{n} \varphi_j 2^{-j}$$

likewise: $U^2\,|u\rangle = \exp(2\pi i \varphi \cdot 2)\,|u\rangle$

$$U^{2^k}\,|u\rangle = \exp(2\pi i \varphi \cdot 2^k)\,|u\rangle \quad \text{for } 0 \leq k \leq n-1$$

This suggests building the following circuit (see next page).

# Phase estimation circuit ($n \geq 1$)



The state $|\varphi\rangle$ is given by: (following the $n=1$ computation)

$$|\varphi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + \exp(2\pi i\varphi)|1\rangle$$

$$\otimes \frac{1}{\sqrt{2}}\big(|0\rangle + \exp(2\pi i\varphi\cdot 2)|1\rangle$$

$$\vdots$$

$$\otimes \frac{1}{\sqrt{2}}\big(|0\rangle + \exp(2\pi i\varphi\cdot 2^{n-1})|1\rangle\Big) \otimes |u\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{z_1 \ldots z_n \in \{0,1\}} \exp\Big(2\pi i\varphi \sum_{k=1}^{n} z_k 2^{k-1}\Big)|z_1 \ldots z_n\rangle \otimes |u\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{z=0}^{2^n-1} \exp(2\pi i\varphi z)\,|z\rangle \otimes |u\rangle$$

Observe finally that

$$\varphi = \sum_{j=1}^{n} \varphi_j 2^{-j} = \frac{1}{2^n} \sum_{j=1}^{n} \varphi_j 2^{n-j} = \frac{1}{2^n} \phi$$

where $\varphi_1 \dots \varphi_n$ is now the binary decomposition of $\phi \in \{0, \dots, 2^{n-1}\}$

$$\text{So } |\psi\rangle = \frac{1}{2^{n/2}} \sum_{z=0}^{2^n-1} \exp\left(\frac{2\pi i z \phi}{2^n}\right) |z\rangle \otimes |u\rangle$$

$$= QFT\,|\phi\rangle \otimes |u\rangle$$

Therefore, the final output of the circuit is $QFT^\dagger(QFT\,|\phi\rangle) \otimes |u\rangle$ $= |\phi\rangle \otimes |u\rangle$, so the measure of the first $n$ qubits gives $|\phi\rangle = |\varphi_1 \dots \varphi_n\rangle$.

(Of course, this only works perfectly under the assumption that the phase $\varphi = \sum_{j=1}^{n} \varphi_j 2^{-j}$.)