Stability of cavity modes in a laser

The aim of this tutorial is to derive the stability criterion for the laser oscillation within the approximation of geometric optics. The plane-wave approach is instructive but too rough for several applications. In the first part, we derive the matrix representation of some optical elements. Next, we discuss periodic optical system using the ray-transfer matrix formalism. We finally apply this results to derive a specific stability criterion of a laser cavity

1 Matrix formulation of geometric optics

We define an optical axis along the x-direction. In this framework, a light beam can be represented by 3 numbers forming a vector (see figure) (y, θ, x). In practice, we are interested in the beam caracteristics at known x-coordinates on the axis, for instance where we have a lens, a mirror, and so on...

1. If the beam travels in free space over a distance d, how, does the beam vector evolve ? Deduce the ray-transfer matrix for free space.

2. Using the same logic, derive the expression of the ray transfer-matrix for a curved mirror with ROC R.

3. The motivated student can try the same approach with a lens of focal f. It won't be useful for next part though.

2 Periodic optical system

Let us imaggine that we have a periodic system, meaning that the ray-transfer matrix is infinitely applied to a system. We therefore want to label our light-vector as a sequence such that :

$$\begin{pmatrix} y_{m+1} \\ \theta_{m+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_m \\ \theta_m \end{pmatrix}$$

1. Show that :

$$\theta_m = \frac{y_{m+1} - Ay_m}{B}$$

$$y_{m+2} = 2by_{m+1} - F^2 y_m \qquad b = \frac{A+D}{2} \qquad F^2 = AD - BC$$

2. Show that $y_m = y_0 h^m$ is a solution, and specify h

3. The previous expression requires the use of Newton's expansion. To avoid this, we define $\varphi = \arccos(b/F)$. Show that in with this notations :

$$y_m = y_0 F^m e^{\pm jm\phi}$$

4. Construct a sinusoidal solution using the previous expansion, ie show that one can write $y_m = y_{max} \sin(m\varphi + \varphi_0)$. What is the condition to have this harmonic trajectory ?

3 The laser cavity

A laser cavity consists in two mirrors of ROC R_1 and R_2 separated by a distance d. Using the previous results, show that the oscillations are stable iff :

$$0 \leq \left(1 + \frac{d}{R_1}\right)\left(1 + \frac{d}{R_2}\right) \leq 1$$