Lecture 2

Grad-Shafranov solutions: toroidal and shaped axisymmetric equilibria

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Content of this lecture

2. Grad-Shafranov solutions: toroidal and shaped axisymmetric equilibria

- The poloidal flux ψ, toroidal field $B_\phi = F/R$, the local pitch q_l and safety factor q
- Straight field line coordinates
- Flux coordinates for equilibrium expansion
- Harmonic and ϵ expansion of the Grad Shafranov Equation
- Solution to the toroidal field
- The equation for the Shafranov shift
- The equation for the penetration of cross section shaping
- Toroidicity and pressure: particle trapping

Recap of Lecture 1

The Grad-Shafranov equation has been derived. It was shown that a tokamak equilibrium can be generated via:
(1) Given profiles in the pressure $P(\psi)$ and the 'current' function $F(\psi)$ with respect to ψ.
(2) The flux ψ on a fixed boundary defined by $(R, Z)_{boundary}$.

With the introduction of flux coordinates, where $\psi = \psi(r)$, with r a minor radius with units of length, we are able to initialise the problem with
(1) $P(r)$, $q(r)$ (via Eq. (2.3). An alternative to $q(r)$ would be to define the toroidal current density J averaged over the flux surface defined by r, since from Eqs. (1.12) and (1.15) we have

$$\left\langle J_\phi \right\rangle(r) = \frac{\langle R \rangle}{\psi'} P' + \left\langle \frac{1}{R} \right\rangle \frac{F}{\psi'} F' \quad \text{where} \quad \langle X \rangle(r) = \frac{\int_0^{2\pi} d\omega \, X \mathcal{J}}{\int_0^{2\pi} d\omega \, \mathcal{J}}$$

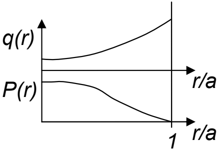
(2) The flux ψ on a fixed boundary defined by $(R(r, \omega), Z(r, \omega))_{boundary}$

Typically, one would verify that the choice of ψ (with units Tesla m^2) at the boundary is consistent with the toroidal magnetic field (known experimentally). Also known, to some extent experimentally is the distribution of current $I_\phi(r)$, and certainly I_ϕ enclosed within the vessel. Note that $\left\langle J_\phi \right\rangle(r)$ is related to the toroidal current $I_\phi(r)$ enclosed within the surface defined by r.

From this, one will then obtain the surfaces in $R(r, \omega), Z(r, \omega)$ on which the field lines and current lines lie (the flux surfaces). Moreover, we can obtain $\boldsymbol{B}(r, \omega)$ and $\boldsymbol{J}(r, \omega)$ etc.

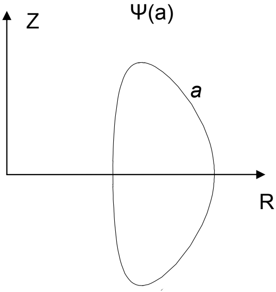
Recap of Lecture 1

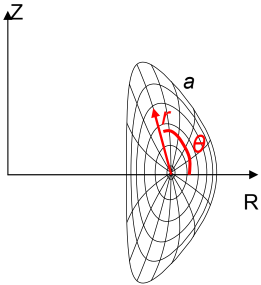
Input



Output

$\Psi'(r)$
$\mathbf{B}(r,\theta)$
$\mathbf{J}(r,\theta)$
etc





Boundary maps out $R(a,\theta)$, $Z(a,\theta)$,

Flux Coordinate representation of $\boldsymbol{B}$

We start with the general coordinate system (r, Θ, ϕ), the system becoming a flux coordinate system by setting $\psi = \psi(r)$, i.e. ψ is independent of poloidal angle.

For such flux coordinates, we may write the full field Eq. (1.10) as

$$\begin{aligned}
\boldsymbol{B} &= B_\phi \boldsymbol{e}_\phi + B_p \boldsymbol{e}_p \\
&= F(r)\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi \\
&= \underbrace{\frac{F(r)}{R(r,\Theta)}}_{B_\phi} \underbrace{R\boldsymbol{\nabla}\phi}_{\boldsymbol{e}_\phi} + \underbrace{\frac{\psi(r)'|\boldsymbol{\nabla}r|}{R(r,\Theta)}}_{B_p} \underbrace{\frac{R(r,\Theta)\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}r}{|\boldsymbol{\nabla}r|}}_{\boldsymbol{e}_p}
\end{aligned} \tag{2.1}$$

where $\psi' = d\psi/dr$ and $\boldsymbol{e}_{\phi,p}$ are unit vectors. Notice that this definition of B_p is consistent with the intuitive definition of the poloidal flux of Eq. (1.11),

$$B_p = \frac{|\boldsymbol{\nabla}r|}{R}\frac{d\psi}{dr} \quad \rightarrow \quad \psi = \int dr \, \frac{RB_p}{|\boldsymbol{\nabla}r|}$$

As a final remark, it is always possible to define $\boldsymbol{B}$ in Clebsch form,

$$\boldsymbol{B} = \boldsymbol{\nabla}\beta \times \boldsymbol{\nabla}\psi = \psi' \boldsymbol{\nabla}\beta \times \boldsymbol{\nabla}r, \quad \beta = \beta(r, \Theta, \phi)$$

As will be seen in the exercises and later when we consider stability problems, the Clebsch form can be a convenient definition when using straight field line coordinates, less so otherwise.

- In this course we will use $\Theta \rightarrow \omega$ for equilibrium expanded coordinates, and $\Theta \rightarrow \theta$ for straight field line coordinates. As we will see, we will adopt the same radial variable for both.

Pitch q_l and q for flux coordinates (r, Θ, ϕ)

Recall that the local field pitch q_l defined by Eq. (1.16) defines the trajectory of the field line on each flux surface ψ or r, hence,

$$q_l = \frac{d\phi}{d\Theta} = \frac{\boldsymbol{B} \cdot \boldsymbol{\nabla}\phi}{\boldsymbol{B} \cdot \boldsymbol{\nabla}\Theta}$$

Noting that the Jacobian of the flux coordinate system (r, Θ, ϕ) is $\mathcal{J}_\Theta^{-1} = (\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}r) \cdot \boldsymbol{\nabla}\Theta$ we see that,

$$q_l = \frac{\boldsymbol{B} \cdot \boldsymbol{\nabla}\phi}{\boldsymbol{B} \cdot \boldsymbol{\nabla}\Theta} = \frac{F/R^2}{\psi'(\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}r) \cdot \boldsymbol{\nabla}\Theta} = \frac{F(r)\mathcal{J}_\Theta}{\psi(r)' R^2} \tag{2.2}$$

The safety factor is the poloidally averaged field line pitch, defined via Eq. (1.17). For general flux coordinates (r, Θ, ϕ) we have that,

$$q(r) = \frac{1}{2\pi} \int_0^{2\pi} q_l(r, \Theta)\, d\Theta = \frac{F(r)}{\psi'} \frac{1}{2\pi} \int_0^{2\pi} d\Theta\, \frac{\mathcal{J}_\Theta}{R(r, \Theta)^2} \tag{2.3}$$

Straight field line flux coordinates (r, θ, ϕ)

We now explore the characteristics of straight field line coordinates (r, θ, ϕ). As explained earlier, these have the property $q_l = q$, i.e. that q_l is independent of poloidal angle $\Theta \to \theta$. Clearly, from Eq. (2.2) we have that the **poloidal dependence in $\mathcal{J}$ enters solely via a proportionality with R^2**,

$$\mathcal{J}_\theta = \frac{q(r)\,\psi' R(r,\theta)^2}{F(r)}. \tag{2.4}$$

We can now identify how ψ' depends on r by consideration of the volume element dv^3 which by definition is

$$dv^3 = \mathcal{J}_\theta\, dr\, d\theta\, d\phi$$

At the magnetic axis $r = 0$, we have that $R(r, \theta) = R_0$ a constant. Infintesimally close to the axis the volume element is $dv^3 = R_0 r\, dr\, d\theta\, d\phi$, and away from $r = 0$, $\mathcal{J}_\theta$ must have poloidal dependence proportional to R^2. These two constraints ensure the volume element and Jacobian are given by

$$dv^3 = \mathcal{J}_\theta\, dr\, d\theta\, d\phi, \quad \mathcal{J}_\theta = \frac{r R(r,\theta)^2}{R_0}.$$

Equating this identity of the Jacobian with Eq. (2.4) we have

$$\psi' = \frac{r}{R_0}\frac{F(r)}{q(r)} \tag{2.5}$$

or rearranging,

$$q(r) = \frac{r}{R_0}\frac{F(r)}{\psi'} \tag{2.6}$$

or from taking the ratio of the field components in Eq. (2.1), $q = r B_\phi |\boldsymbol{\nabla} r| / (R_0 B_p)$.

Fourier expansion of equilibrium

- Notice that the identification of the straight field line system has not yet helped us solve the equilibrium problem, in particular the location of the contours of constant ψ in the R-Z cross section. One way to do that is to Fourier expand R and Z over angle Θ
- Substitution the Fourier series into the Grad-Shafranov operator (1.13) and equation (1.15) will identify the equations for the coefficients.
- We will see that the up-down symmetric series of Eqs. (1.24) and (1.25) when substituted into Grad-Shafranov equation allows all the equations for the coefficients to decouple from each other.
- We expand the coefficients in smallness parameter $\epsilon = r/R_0$, the local inverse aspect ratio (note, near the magnetic axis this parameter is small even in a spherical tokamak). First let us assume only intuitive Shafranov-displaced circular surfaces:

$$R(r,\Theta) = R_0\left(1 + \varepsilon\,\epsilon\cos\Theta - \varepsilon^2\frac{\Delta(r)}{R_0} + O(\varepsilon^3)\right)$$
$$Z(r,\Theta) = R_0\left(\varepsilon\,\epsilon\sin\Theta + O(\varepsilon^3)\right),$$

This particular expansion in R and Z can now easily be shown to be inconsistent with straight field line coordinate system. The Jacobian (see notes pages Lecture 1, and exercises) is:

$$\mathcal{J}_\Theta = R\left(\frac{\partial R}{\partial r}\frac{\partial Z}{\partial \Theta} - \frac{\partial R}{\partial \Theta}\frac{\partial Z}{\partial r}\right)$$

Substitution of R and Z above into $\mathcal{J}_\Theta$ yields

$$\mathcal{J}_\Theta = rR_0\left\{1 + \varepsilon(\epsilon - \Delta')\cos\Theta + O(\varepsilon^2)\right\}$$

which is **not** proportional to $R(r,\Theta)^2$, i.e. the flux coordinates of this aspect ratio expanded equilibrium are not straight field line coordinates. For this equilibrium expansion problem, we use $\Theta \to \omega$ henceforth, and denote the Jacobian $\mathcal{J}_\omega$.

Equilibrium expanded coordinates (r, ω, ϕ)

Stability problems are most easily treated with straight field line coordinates (reasons given in later lectures), but to evaluate dispersion relations, growth rates etc, coefficients in the stability equations will depend on toroidicty (or shaping), and these effects will need to be converted to the straight field line coordinates deployed in the stability problem.

- The conversion problem is simplified by ensuring the radial variable r is the same in both systems, and hence the dependence of the flux ψ on r.
- We ensure the safety factor profile $q(r)$ is the same for systems, and thus from Eq. (2.5) $F(r)$ is the same for both.

Although $F(r)$ and $\psi(r)$ are the same, it should be reminded that B_ϕ and B_p depend on poloidal angle, and hence are distributed differently across the coordinate systems.

Equations (2.1), (2.2), (2.3), (2.5) and (2.6) are valid for both systems. An important constraint for the equilibrium expanded system is formed by equating Eqs. (2.3) with (2.6), which gives:

$$\frac{r}{R_0} = \frac{1}{2\pi} \int_0^{2\pi} d\omega \, \frac{\mathcal{J}_\omega}{R^2} \qquad (2.7)$$

This constraint enforces on the equilibrium expanded system (r, ω, ϕ) the relations of Eqs. (2.5) or (2.6), which were obtained intuitively from the straight field line system (r, θ, ϕ).

Coordinate transformation

Finally, with the radial variable being the same, equating the volume elements of each coordinate system reveals a simple coordinate transformation (see exercise),

$$\frac{d\theta}{d\omega} = \frac{\mathcal{J}_\omega}{\mathcal{J}_\theta} \qquad (2.8)$$

Expansion of equilibrium

Plasma shaping is added to the circular shifted expansion in an intuitive way, by adding an even Fourier harmonic to R and subtracting an odd Fourier harmonic to Z. The distance of the flux surface from the magnetic axis, averaged over poloidal angle, is roughly maintained to be r.

$$R(r,\omega) = R_0\left(1 + \varepsilon\,\epsilon\cos\omega - \varepsilon^2\frac{\Delta(r)}{R_0} + \varepsilon^2\sum_{m=2}^{\infty}\frac{S_m(r)}{R_0}\cos(m-1)\omega + \varepsilon^3\frac{\mathcal{P}(r)}{R_0}\cos\omega + O(\varepsilon^4)\right) \tag{2.9}$$

$$Z(r,\omega) = R_0\left(\varepsilon\,\epsilon\sin\omega - \varepsilon^2\sum_{m=2}^{\infty}\frac{S_m(r)}{R_0}\sin(m-1)\omega + \varepsilon^3\frac{\mathcal{P}(r)}{R_0}\sin\omega + O(\varepsilon^4)\right), \tag{2.10}$$

We carry an artificial tag, ε, which is used to denote the size of each term relative to the inverse aspect ratio. This approach is very useful for algebraic manipulation and minimisation, computational algebra or otherwise. There is an assumption on the size of shaping coefficients relative to ϵ. We obtain (exercises).

$$\mathcal{J}_\omega = rR_0\left\{1 + \varepsilon(\epsilon - \Delta')\cos\omega + \varepsilon\sum_{m=2}^{\infty}\left(S'_m - (m-1)\frac{S_m}{r}\right)\cos(m\omega) + O(\varepsilon^2)\right\} \tag{2.11}$$

$$g_{r,\omega} = r\left\{\varepsilon\Delta'\sin\omega - \varepsilon\sum_{m=2}^{\infty}\left(S'_m + (m-1)\frac{S_m}{r}\right)\sin(m\omega) + O(\varepsilon^2)\right\} \tag{2.12}$$

$$g_{\omega,\omega} = r^2\left\{1 - \varepsilon 2\sum_{m=2}^{\infty}(m-1)\frac{S_m}{r}\cos(m\omega) + O(\varepsilon^2)\right\} \tag{2.13}$$

Although not needed for this lecture, stability problems sometimes require the metric evaluated at higher. For this, $\mathcal{P}(r)$ is retained as an unknown, then the constraint of Eq. (2.7) yields $\mathcal{P}(r)$ in terms of the coefficients, which is then substituted back into the geometric tensor at higher order.

$$\frac{\mathcal{P}(r)}{R_0} = \epsilon\left(\frac{\epsilon^2}{8} + \frac{\Delta(r)}{2R_0}\right) + \sum_{m=2}^{\infty}\frac{1-m}{2\epsilon}\left(\frac{S_m(r)}{R_0}\right)^2$$

Notes

The exercises from week 1 use

$$\mathcal{J}_\omega = R\left(\frac{\partial R}{\partial r}\frac{\partial Z}{\partial \omega} - \frac{\partial R}{\partial \omega}\frac{\partial Z}{\partial r}\right), \quad g_{r,\omega} = \frac{\partial Z}{\partial r}\frac{\partial Z}{\partial \omega} + \frac{\partial R}{\partial r}\frac{\partial R}{\partial \omega}, \quad g_{\omega,\omega} = \left(\frac{\partial Z}{\partial \omega}\right)^2 + \left(\frac{\partial R}{\partial \omega}\right)^2$$

The expression for Jacobian can be calculated to the next order, with $\mathcal{P}(r)$ as an unknown (see e.g. J. P. Graves, PPCF **55**, 074009 (2013)):

$$\mathcal{J}_\omega = rR_0(J_0 + J_1 + J_2),$$

with

$$\begin{aligned}
J_0 &= 1 \\
J_1 &= \left(\epsilon - \Delta'\right)\cos\omega + \sum_m \left((1-m)\frac{S_m}{r} + S_m'\right)\cos m\omega \\
J_2 &= -\frac{\Delta}{R_0} + \frac{S_m}{R_0}\cos(m-1)\omega - \cos\omega\left[\epsilon\Delta'\cos\omega + \left(\frac{(m-1)S_m}{R_0} - \epsilon S_m'\right)\cos m\omega\right] \\
&\quad \frac{-\mathcal{P} - r\mathcal{P}'}{\epsilon R_0} - (m-1)S_m\left(S_m' - \Delta'\cos(m-1)\omega\right)
\end{aligned}$$

Substituting this and the expansion of R into the constraint of Eq. (2.7) yields,

$$\frac{\mathcal{P}(r)}{R_0} = \epsilon\left(\frac{\epsilon^2}{8} + \frac{\Delta(r)}{2R_0}\right) + \sum_{m=2}^{\infty}\frac{1-m}{2\epsilon}\left(\frac{S_m(r)}{R_0}\right)^2.$$

The geometric tensor is then known to higher order, e.g.

$$J_2 = -\frac{\epsilon}{2}[\epsilon + \Delta'(2+\cos 2\omega)] - 2\frac{\Delta}{R_0} + \sum_m\left[\epsilon S_m' + \frac{S_m}{R_0}(1-m)\right]\cos\omega\cos m\omega + \sum_m \frac{S_m}{R_0}\left[1 + (m-1)\frac{\Delta'}{\epsilon}\right]\cos(m-1)\omega$$

Harmonic and ϵ expansion of the Grad Shafranov Equation

Depositing these ε expansion terms into the Grad-Shafranov equation of Eqs. (1.22) and (1.15) yields

$$\frac{\psi'^2}{r}\left\{1+\varepsilon\left(r\Delta''+\Delta'-\frac{r}{R_0}\right)\cos\omega-\varepsilon\sum_{m=2}^{\infty}\left(rS''_m+S'_m+(1-m^2)\frac{S_m}{r}\right)\cos(m\omega)+O(\varepsilon^2)\right\}+$$

$$\psi'\,\psi''\left\{1+\varepsilon 2\Delta'\cos\omega-\varepsilon 2\sum_{m=2}^{\infty}S'_m\cos(m\omega)+O(\varepsilon^2)\right\}+R_0^2\left(1+\varepsilon 2\frac{r}{R_0}\cos\omega+O(\varepsilon^2)\right)P'+FF'=0 \quad (2.14)$$

So, gathering Fourier components, we clearly have

$$\cos(0\omega):\quad \left[\frac{1}{2r^2}\left(r^2\psi'^2\right)'+R_0^2P'+FF'\right][1+O(\varepsilon^2)]=0, \quad (2.15)$$

$$\cos(\omega):\quad \left[\Delta''+\left(2\frac{\psi''}{\psi'}+\frac{1}{r}\right)\Delta'-\frac{1}{R_0}+2\frac{rR_0P'}{(\psi')^2}\right][1+O(\varepsilon)]=0, \quad (2.16)$$

$$\cos(m\omega):\quad \left[S''_m+\left(2\frac{\psi''}{\psi'}+\frac{1}{r}\right)S'_m+\frac{1-m^2}{r^2}S_m\right][1+O(\varepsilon)]=0. \quad (2.17)$$

- Eq. (2.15) describes the balances of forces by the plasma pressure (∇P) and the magnetic pressure ($\nabla B^2/2$), and describes how the poloidal and toroidal fields prevent the plasma from expanding in the ∇r direction.
- Eq. (2.16) yields the radial equation for the Shafranov shift, and expresses how the poloidal field prevents the plasma pressure from expanding the plasma in the ∇R direction.
- Eq. (2.17) yields the radial equation for the shaping. The shaping profiles depend only on the shaping at the boundary, and (we will see) on the q-profile. It has no dependence on the pressure at this order.

The full toroidal field from ∇r equation

We now solve Eq. (2.15), which is an equation for $F(r)$. This of course provides us with the solution to the toroidal magnetic field, $B_\phi = F(r)/R$. An order expansion of Eq. (2.15) shows us that we write

$$F = R_0 B_0[1 + \varepsilon^2 F_2(r) + O(\varepsilon^4)], \tag{2.18}$$

thus the solution to Eq. (2.15) provides us with the weak correction $F_2(r)$.

Eq. (2.15) depends on operations on ψ', which from Eq. (2.5) we have

$$\frac{\left(r^2\psi'^2\right)'}{2r^2} = \frac{(\psi')^2}{r}\left[2 - s + \frac{rF'}{F}\right] \quad \text{with} \quad s = \frac{r}{q}\frac{dq}{dr}$$

$$\psi' = \frac{rB_0}{q}[1 + O(\varepsilon^2)], \quad \text{and thus} \quad \frac{\left(r^2\psi'^2\right)'}{2r^2} = \left(\frac{rB_0}{q}\right)^2 \frac{1}{r}\left[2 - s + O(\varepsilon^2)\right] \tag{2.19}$$

Hence, substituting F and Eq. (2.19) into Eq. (2.15) we have

$$\left(\varepsilon\frac{r}{R_0}\right)^2\left(\frac{2-s}{q^2}\right)\left(1 + O(\varepsilon^2)\right)^2 + \varepsilon^2\frac{rP'}{B_0^2} + \left[1 + \left(\varepsilon\frac{r}{R_0 q}\right)^2\right]\left(1 + O(\varepsilon^2)\right)\varepsilon^2 rF_2' = 0,$$

Taking leading order terms (i.e. dropping $O(\varepsilon^4)$ terms, we have on integrating:

$$\varepsilon^2 F_2 = -\varepsilon^2\frac{P}{B_0^2} - \varepsilon^2\int_0^r \frac{dr\ r}{R_0^2}\left(\frac{2-s}{q^2}\right) + O(\varepsilon^4) \tag{2.20}$$

Note that the orders above assume that $P/B_0^2 \sim \varepsilon^2$, which is conventional ordering. However Eqs. (2.18) and (2.20) are valid even when $P/B_0^2 \sim \varepsilon$, although F_2 would be renamed F_1 and the ε tags would have to be modified.

The effect of pressure on B_ϕ and B

The toroidal field $B_\phi \boldsymbol{e}_\phi = F\boldsymbol{e}_\phi / R$ has now been fully described. Need still to obtain the poloidal field, and thus the total field, and field strength. Now from Eq. (1.10) the poloidal field

$$\boldsymbol{B}_p = \psi' \boldsymbol{\nabla}\phi \times \boldsymbol{\nabla} r = \psi' |\boldsymbol{\nabla} r||\boldsymbol{\nabla}\phi| \boldsymbol{e}_p \quad \text{where} \quad \boldsymbol{e}_p = \frac{\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla} r}{|\boldsymbol{\nabla} r||\boldsymbol{\nabla}\phi|}$$

Here, clearly, $\boldsymbol{e}_p$ is the unit vector pointing in the poloidal direction. From Eqs.(1.20) and (2.19), and noting that $|\boldsymbol{\nabla}\phi|^2 = 1/R^2$ and $|\boldsymbol{\nabla} r|^2 = g_{\omega,\omega} R^2 / \mathcal{J}_\omega^2$, we have:

$$\boldsymbol{B}_p = \psi' \frac{\sqrt{g_{\omega,\omega}} \boldsymbol{e}_p}{\mathcal{J}_\omega} = B_0 \left[\left(\frac{\varepsilon r}{R_0 q} \right) \frac{1}{1 + \varepsilon(\epsilon - \Delta') \cos\omega + \varepsilon \sum_{m=2}^{\infty} S'_m \cos m\omega} + O(\varepsilon^3) \right] \boldsymbol{e}_p . \tag{2.21}$$

For many applications, it of great interest to evaluate the magnetic field strength $|\boldsymbol{B}|$, or indeed $\boldsymbol{B}^2$. Since $\boldsymbol{e}_p$ is perpendicular to $\boldsymbol{e}_\phi$ we have $B^2 = B_\phi^2 + \boldsymbol{B_p}^2$:

$$B^2 = B_0^2 \left[\left(\frac{1 + F_2}{1 + \epsilon\cos\omega - \Delta/R_0 + \sum_{m=2}^{\infty} (S_m/R_0) \cos(m-1)\omega} \right)^2 + \frac{\epsilon^2}{q^2} \left(\frac{1}{1 + (\epsilon - \Delta')\cos\omega + \sum_{m=2}^{\infty} S'_m \cos m\omega} \right)^2 + O(\varepsilon^4) \right] \tag{2.22}$$

We now see that the field strength is affected by the plasma pressure in two ways.
(1) Pressure reduces the toroidal field through F_2. This is known as the diamagnetic effect of the plasma pressure.
(2) Furthermore, we will see that Δ' is governed by the plasma pressure, and this in turn strongly influences the poloidal field. It is clear that this enters through the $1/\mathcal{J}_\omega$ dependence of B_p, which in turn enters through the local field line pitch $d\phi/d\omega = F\mathcal{J}_\omega / (R^2 \psi')$.

The ∇R equation: Shafranov Shift

So, here we consider the force balance equation with $\cos\omega$ dependence, i.e. Eq. (2.16). This equation can be integrated once, to obtain

$$\Delta' = \frac{1}{R_0 r \psi'^2}\left[-2R_0^2\int_0^r dr\, r^2 P' + \int_0^r dr\, r\psi'^2\right]$$

Now, recalling that $\psi' = rF/(qR_0)$ we obtain the convenient expression

$$\Delta'(r) = \frac{r}{R_0}\left[\beta_p(r) + \frac{l_i(r)}{2}\right] \tag{2.23}$$

where β_p is the local poloidal beta

$$\beta_p(r) = -2\frac{R_0^4 q^2}{F^2 r^4}\int_0^r dr\, r^2 P' \approx \frac{2}{\left\langle B_p^2\right\rangle}\left[\langle P\rangle - P(r)\right] \quad \text{with} \quad F \approx R_0 B_0 \tag{2.24}$$

where $\langle X\rangle = V(r)^{-1}\int_0^{V(r)} dV X$ is average of X within the volume defined by r. Also, l_i is the local internal inductance

$$l_i(r) = 2\frac{q^2}{F^2 r^4}\int_0^r dr\,\frac{r^3F^2}{q^2} \approx 2\frac{q^2}{r^4}\int_0^r dr\,\frac{r^3}{q^2} \approx \frac{4}{I_p(r)^2 R_0}\int_0^{V(r)} \frac{1}{2}B_p^2\, dV. \tag{2.25}$$

where I_p is the plasma current.
Equation (2.23) describes $\boldsymbol{J}\times\boldsymbol{B} = \nabla P$ force balance in the ∇R direction. The first term on the right attempts to increase the plasma volume. Since $V \approx 2\pi^2 r^2 R$, the volume of a torus can of course can be increased by increasing R. This force is known as the Tyre Tube Force, since the air pressure in a toroidal inner tube similarly expands the major radius of the tyre.

The ∇R equation: Shafranov Shift

The second term on the right of Eq. (2.23) is due to the toroidal current. It turns out the origin for this outward force, known as the Hoop Force, in the ∇R direction, is analogous to the Tyre Tube Force, but where the plasma pressure is replaced by minus the poloidal magnetic pressure. To see this, one has to consider the two relevant quantities ψ and P, both of which are constants on the flux surface, and thus have the same value on the inboard or outboard side. Since ψ passes a smaller area on the inboard side than the outboard side, then the $B_p(in) > B_p(out)$ (this is clear from the $1/R$ dependence in B_p, though the effect is reduced for increased Shafranov shift). The outward Hoop Force is

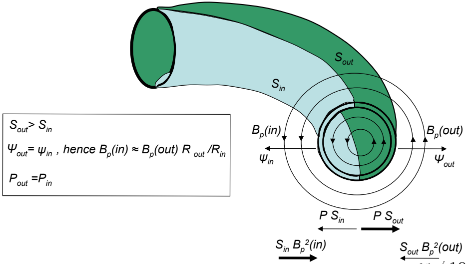
$$F_{Hoop} \sim [B_p(in)^2 S_{in} - B_p(out)^2 S_{out}]\boldsymbol{e}_R,$$

where the quadratic dependence of B_p wins over the inverse dependence of S. Similarly, the Tyre Tube Force can be written

$$F_{Tyre} = -P(S_{in} - S_{out})\boldsymbol{e}_R,$$

whereby, as mentioned earlier, the force occurs because $S_{in} < S_{out}$.

The balancing of the force on the left hand side of (2.23), i.e. Δ', is caused by the compression of the flux surfaces, i.e. the compression of the poloidal flux on the LFS of the torus. Clearly the compression is measured by the Shafranov shift displacement Δ.



Experimental Shafranov Shift

By inspection of β_p and l_i is is clear that Δ' and Δ are monotonically increasing functions of r. Our definition of the Shafranov shift, Δ, is the shift relative to the magnetic axis B_0, which is located at $R = R_0$. Beware! This is not the definition understood by experimentalists!

For an experimentalist, the real world starts at the vessel wall, or the plasma edge. So any shift, due to toroidal effects, would be relative to the plasma edge. Defining $R_{out}(a)$ as the plasma edge (a) on the LFS, and $R_{in}(a)$ as the plasma edge on the HFS, the Shafranov shift at a particular flux surface r is typically defined as the difference between the 'centres' of these surfaces:

$$\Delta_{exp}(r) = [R_{out}(r) + R_{in}(r)]/2 - [R_{out}(a) + R_{in}(a)]/2$$

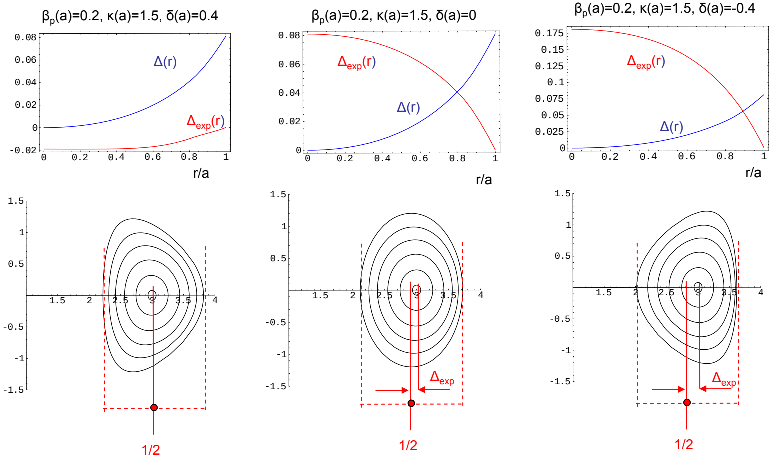
Hence Δ_{exp} is maximum at the magnetic axis, i.e. the opposite of the definition of the theoretician. Nevertheless, experimentalists should also beware. Since $R_{out} = R(r, \omega = 0)$ and $R_{in} = R(r, \omega = \pi)$, then referring to Eqs. (1.24) and (1.28) we have:

$$\Delta_{exp}(r) = \Delta(a) - \Delta(r) + \frac{r\delta(r)}{4} - \frac{a\delta(a)}{4} \quad \text{and especially} \quad \Delta_{exp}(0) = \Delta(a) - \frac{a\delta(a)}{4}$$

A measurement of enhanced Δ_{exp} might lead one to believe that the configuration might be more resistant to instability, since e.g. it is known that pressure gradients drive various instabilities as well as the Shafranov shift. So, a large Shafranov shift would indicate that large pressure gradients have been estabilished, and the very existance of large pressure gradients would imply that MHD instabilities must have been tamed. However, one can also enhance $\Delta_{exp}(r)$ via e.g. negative triangularity and lower pressure gradients.

Hence, to summarise, the Shafranov shift is a very important quantity. It describes the primary effect of toroidicity in a tokamak. Many of the stability properties derived in the straight cylindrical approximation of a torus are entirely cancelled by toroidal effects. For such instabilities, the remaining stability criteria are due to toroidicity (and shaping), and sometimes the criteria can be written explicitly in terms of the Shafranov shift.

Shafranov shift with negative and positive triangularity



Equation for shaping coefficients

The shaping coefficients are given by the solution of the $\cos(m\omega)$ terms in the Grad-Shafranov equation, i.e. Eq. (2.17). Substituting Eq. (2.19), and ignoring small corrections in F_2', we have,

$$r^2 S_m'' + [3 - 2s(r)] r S_m' + (1 - m^2) S_m = 0, \tag{2.26}$$

so we see that the penetration of the shaping into the core, from the defined shaping at the boundary, depends only on the magnetic shear (to leading order). If the magnetic shear is vanishingly small, one simply has

$$S_m(r) = S_m(a) \left(\frac{r}{a}\right)^{m-1}$$

So that, from Eq. (1.28), one has for a flat q-profile:

$$\kappa(r) = \kappa(a) \quad \text{and} \quad \delta(r) = \delta(a) \frac{r}{a},$$

i.e. elongation remains constant from the edge to the magnetic axis, while the triangularity drops off linearly towards the core, and thus vanishes at the magnetic axis.

Magnetic shear modifies the penetration of the shaping into the core. An increase in the magnetic shear reduces the shaping in the core, while a decrease in shear increases the shaping. For an advanced scenario, negative magnetic shear actually increases κ and δ inside a region of negative shear. This can be seen by taking

$$q(r) = 1 - \Delta q \left[1 - \left(\frac{r}{r_1}\right)^2\right] \quad \text{and thus} \quad s(r) = 2 \frac{\Delta q}{q(r)} \left(\frac{r}{r_1}\right)^2.$$

Equation for shaping coefficients

Giving exactly,

$$S_m(r) = S_m(a)\left(\frac{r}{a}\right)^{m-1} \frac{q(r)s(r) + 2(1-\Delta q)(1+m)/(m-1)}{q(a)s(a) + 2(1-\Delta q)(1+m)/(m-1)}$$

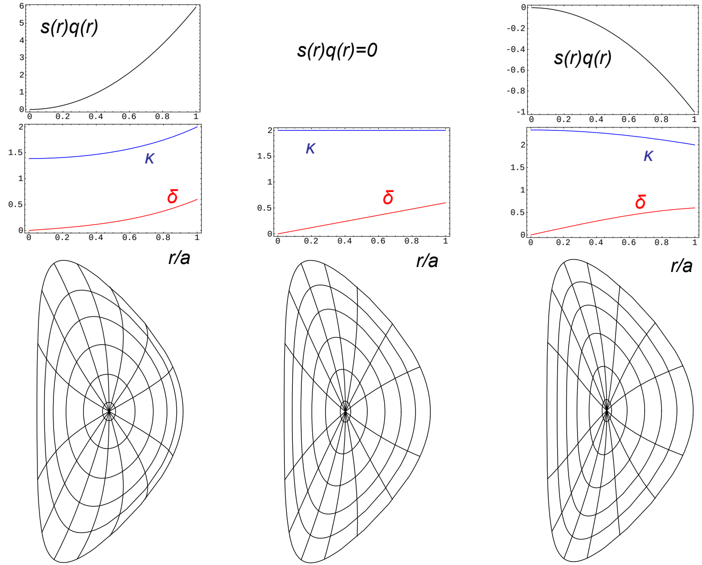
For $\Delta q \ll 1$ one then obtains the useful expressions,

$$\kappa(r) = \kappa(a) - \frac{[\kappa(a)^2 - 1][q(a)s(a) - q(r)s(r)]}{12 + [1+\kappa(a)]q(a)s(a) - [\kappa(a)-1]q(r)s(r)} \quad \text{and} \quad \delta(r) = \delta(a)\left(\frac{r}{a}\right)\frac{4 + q(r)s(r)}{4 + q(a)s(a)}$$

These expressions are almost exact for a q-profile with a quadratic dependence in r. However, since these expressions are written in terms of $q(r)$ and $s(r)$, they might be used with some confidence for wider classes of safety factor. Certainly, these expressions describe the penetration of the shape from an arbitrary surface, a (not necessarily the plasma edge) into a surface inside, denoted by r, where the q-profile in the region $\{r, a\}$ is locally quadratic.

The penetration of the shaping into the core region, from the boundary, is important from the stability point of view. Linear MHD stability in tokamaks is sensitive to the shaping (and Shafranov shift) in the region where the particular instability exists, and especially on rational surfaces. Rational surfaces and MHD instabilities will be considered in the remainder of this course.

Shaped Plasma Cross Section



Notes

Components of $\boldsymbol{B_p}$

We know that the poloidal field is perpendicular to the toroidal field. Moreover, since the ω direction is not the poloidal direction, then if we attempt to write the poloidal field in terms of the ω direction, there has to be a correction factor pointing in the r (outwards from the flux surface) direction. However, we exploit the fact that we know that the true poloidal direction is perpendicular to r. Let,

$$\boldsymbol{e}_p \equiv \frac{\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}r}{|\boldsymbol{\nabla}r||\boldsymbol{\nabla}\phi|} = A\boldsymbol{\nabla}\omega + B\boldsymbol{\nabla}r$$

Dotting this with $\boldsymbol{\nabla}r$, we note that the LHS vanishes, hence we may rearrange to obtain

$$B = -A\frac{\boldsymbol{\nabla}r \cdot \boldsymbol{\nabla}\omega}{|\boldsymbol{\nabla}r|^2} = A\frac{g_{r,\omega}}{g_{\omega,\omega}}.$$

We still need to obtain A. However, since $\boldsymbol{e}_p$ is a unit vector, we must have

$$\boldsymbol{e}_p = \frac{\boldsymbol{\nabla}\omega + \frac{g_{r,\omega}}{g_{\omega,\omega}}\boldsymbol{\nabla}r}{\sqrt{\boldsymbol{\nabla}\omega^2 + 2\frac{g_{r,\omega}}{g_{\omega,\omega}}\boldsymbol{\nabla}\omega \cdot \boldsymbol{\nabla}r + \left(\frac{g_{r,\omega}}{g_{\omega,\omega}}\right)^2 \boldsymbol{\nabla}r^2}}.$$

Noting that $\boldsymbol{\nabla}\omega = |\boldsymbol{\nabla}\omega|\boldsymbol{e}_\omega$ and $\boldsymbol{\nabla}r = |\boldsymbol{\nabla}r|\boldsymbol{e}_r$, then from Eq. (1.20) we can rearrange to obtain

$$\boldsymbol{e}_p = \frac{\boldsymbol{e}_\omega + \left(\frac{g_{r,\omega}^2}{g_{\omega,\omega}\,g_{r,r}}\right)^{1/2}\boldsymbol{e}_r}{\left(1 - \frac{g_{r,\omega}^2}{g_{\omega,\omega}\,g_{r,r}}\right)^{1/2}}$$

Notes

Now, $|\boldsymbol{\nabla}\omega|$ which appears in $g_{r,r}$ has yet to be evaluated. One finds that

$$g_{r,r} = \left(\frac{\partial Z}{\partial r}\right)^2 + \left(\frac{\partial R}{\partial r}\right)^2 = 1 + O(\varepsilon^2) = 1 - \varepsilon\left(2\Delta'\cos\omega + 2\sum_{m=2}^{\infty} S'_m \cos m\omega\right) + O(\varepsilon^2)$$

which gives,

$$\sqrt{\frac{g_{r,\omega}^2}{g_{\omega,\omega}g_{r,r}}} = \varepsilon\Delta'\sin\omega - \varepsilon\sum_{m=2}^{\infty}\left(S'_m + (m-1)\frac{S_m}{r}\right)\sin m\omega + O(\varepsilon^2),$$

and thus,

$$\boldsymbol{e}_p = \frac{\boldsymbol{e}_\omega + \left[\Delta'\sin\omega - \sum_{m=2}^{\infty}\left(S'_m + (m-1)\frac{S_m}{r}\right)\sin m\omega\right]\boldsymbol{e}_r}{\left(1 - \left[\Delta'\sin\omega - \sum_{m=2}^{\infty}\left(S'_m + (m-1)\frac{S_m}{r}\right)\sin m\omega\right]^2\right)^{1/2}} + O(\varepsilon^2).$$

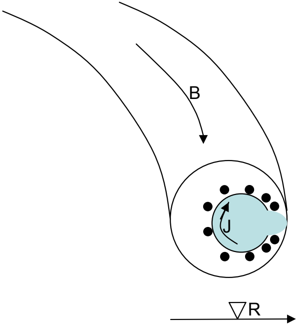
But, noting that the denominator should be replaced by unity at this order, because $\sqrt{1-\varepsilon^2} = \varepsilon^2/2 + ..$, so that

$$\boldsymbol{e}_p = \boldsymbol{e}_\omega + \left[\Delta'\sin\omega - \sum_{m=2}^{\infty}\left(S'_m + (m-1)\frac{S_m}{r}\right)\sin m\omega\right]\boldsymbol{e}_r + O(\varepsilon^2).$$

That the magnetic field appears to point partially in the radial direction seems, on first thought to be wrong, since we know that r is a flux coordinate describing the surface of constant P. In fact, there is not a net radial field. Indeed, $\boldsymbol{e}_\omega$ is not aligned tangentially to $\boldsymbol{e}_p$, which itself is tangent to the poloidal boundary. This means that relative to $\boldsymbol{e}_p$ it turns out that $\boldsymbol{e}_\omega$ has a small component in the $\boldsymbol{e}_r$ direction (recall that the coordinates are not orthogonal). The role of the radial field $\boldsymbol{B}_p \cdot \boldsymbol{e}_r$ is to cancel this out.

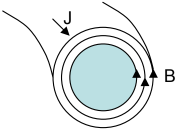
Importance of Poloidal and Toroidal Fields

TORUS WITH PURE TOROIDAL B

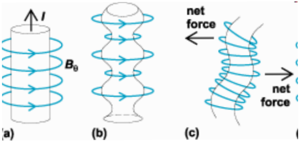


Field lines just slip around flux surface in response to a R-force due to imperfect conductor at wall. Flux cannot be trapped. Applied external field cannot generate J^B in R direction.

TORUS WITH PURE POLOIDAL B



An idealised equilibrium can be generated, Since poloidal flux can be compressed, and Field lines do not displace. Can also use vertical field to create Radial force. But any displacement Of perfect equilibrium, and we have disaster, i.e. very poor MHD stability properties (e.g. sausage).



The Necessity of Vertical Field Coils

Within the plasma, the compression of the magnetic field compensates the Tyre Tube Force and the Hoop Force. At the plasma edge ($r = a$) force balance has to be manufactured by engineering. The poloidal magnetic field at the plasma edge is (see earlier total field and poloidal field expansion):

$$B_p = \left. \frac{\psi' g^{1/2}_{\omega,\omega}}{\mathcal{J}_\omega} \right|_{r=a} = \frac{\psi'(a)}{R_0} \left[1 + [\Delta'(a) - \epsilon(a)] \cos\omega - \sum_{m=2}^{\infty} S'_m(a) \cos m\omega \right]. \tag{2.27}$$

This value must be matched on the vacuum side of the plasma-vacuum interface. The vacuum poloidal field can be established from $J_\phi = 0$, i.e. from Eq. (1.12), $\Delta^*\psi = 0$ for $r > a$. It is possible to integrate $\Delta^*\psi = 0$ outwards using Eq. (2.27) as an initial point. The asymptotic value for B_p for increasing r is proportional to $\cos\omega$. Hence $\boldsymbol{B}_p = B_p \boldsymbol{e}_p$ for large r is orientated vertically.

In practice (for reasonable plasma pulse times) it is necessary to manufacture this vertical field $\boldsymbol{B}_{vert}$ via vertical field coils. The effect on the plasma edge should be intuitive: a force is generated in the $-\boldsymbol{e}_R$ direction via the cross product of the vertical field with the toroidal plasma current, i.e. $F_R \boldsymbol{e}_R = \boldsymbol{J}_\phi \times \boldsymbol{B}_{vert}$

If a conducting wall is placed close to the plasma edge, the poloidal flux will compress between the plasma edge and the wall, providing the necessary force balance without vertical field coils. However, the vessel wall will not be a perfect conductor, and as such, the poloidal flux can only remain compressed for a skin time, which is typically short compared to experimental times of interest. Tokamaks therefore are designed with vertical field coils, and usually they have a fairly close fitting wall.

Toroidicity and pressure effects on single particles

A single particle in a tokamak is considered to be trapped if it cannot circulate around the entire poloidal cross section. The standard definition of a point in phase space where the particle is trapped is given by

$$v_{\|} = 0.$$

Now from elementary theory, we assume that the kinetic energy $E = v_{\|}^2 + v_{\perp}^2$ is conserved over the trajectory of the particle (constant E assumes there is no time varying electric field). Moreover, the magnetic moment $\mu = v_{\perp}^2/(2B)$ is an (adiabatic) invariant. Consequently the poloidal variation of $v_{\|}$ is given by,

$$v_{\|}(r,\omega) = 2\sqrt{E - \mu B(r,\omega)} \tag{2.28}$$

For a low energy particle, which is strongly tied to a flux surface r (so that the banana width is small), the trapping angle ω_t, if it exists, is defined simply by

$$B(r,\omega_t) = E/\mu.$$

The condition for whether a particle is trapped or passing is determined by the location of ω where the field is at a maximum. A trapped particle requires:

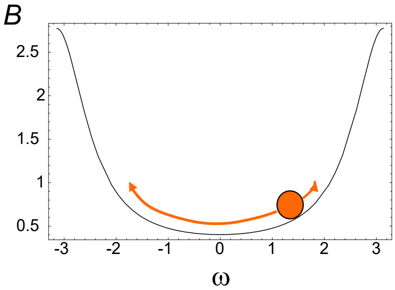
$$B_{\text{max}} = B(r,\omega_{\text{max}}) > E/\mu$$

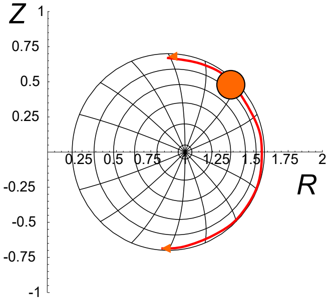
If this state exists, then particles will 'roll around' in a magnetic well.
Ignoring the effects of shaping and pressure in the poloidal field, one simply finds that $B \propto 1/R$, and hence B increases monotonically with ω up to $\omega_{\text{max}} = \pi$. However, in spherical tokamaks, the poloidal field is large (ϵ isn't small), and moreover, pressure gradients are large. Consequently, B is not generally a monotonic function of ω, and resultingly, special (tear-drop) trapped particles are generated.

Application: particle trapping and plasma pressure

Field strength plotted as a function of ω at constant r $(= 0.7)$. Assumed spherical tokamak $B_0 = 1\mathrm{T}$, $R_0 = 1$, $a = 0.7$, with circular cross section ($\kappa(a) = 1$, or $S_2(a) = 0$ and $\delta(a) = 0$, or $S_3(a) = 0$). Chosen monotonic q-profile, and monotonic pressure profile reveal large Shafranov shift through internal inductance (see later). The plasma pressure was taken to be zero.

It is seen that the well in the magnetic field is conventional, meaning that the minimum in B is at $\omega = 0$, and rises monotonically with respect to $\pm\omega$. Consequently, conventional banana trapped particles are to be expected (as well as circulating particles).



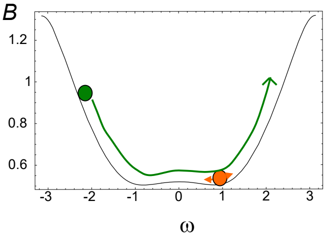


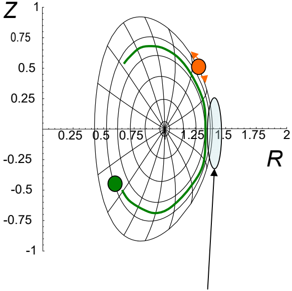
Application: particle trapping and plasma pressure

Again plot field strength as a function of ω at constant r $(= 0.7)$. Assume spherical tokamak $B_0 = 1\text{T}$, $R_0 = 1$, $a = 0.7$, with typical shaped cross section ($\kappa(a) = 1.9$, or $S_2(a) = -0.22$ and $\delta(a) = 0.2$, or $S_3(a) = 0.035$). Monotonic q-profile, and poloidal beta $\beta_p(a) = 0.4$.

The dependence of Magnetic field with ω is unconventional. However, such a dependence is a normal occurrence in a spherical tokamak. There are two minima in the field, each located between $\pi/4 < |\omega| < \pi/2$. Given sufficiently small initial parallel velocity, particles can be trapped in this region. Conventional banana orbits also exist, but the bounce tips of such particles cannot reside on the local magnetic hill!

The local magnetic wells occur because the toroidal field is reduced globally by the pressure (through the diamagnetic effect F_2, and because the poloidal field is generally enhanced because of large ϵ near the edge of a spherical tokamak. The poloidal field can be locally enhanced due to the local shear $q_l = d\phi/d\omega$ i.e. the denominator of B_p is strongly reduced via Δ' at $|\omega| = \pi/2$, and through S_2' at $|\omega| = \pi/4$.





Conventional deeply trapped particles cannot entirely reside in this region

Tear-Drop Orbits in Spherical Tokamaks

The radial excursion of a trapped (or passing) particle can be accurately determined in terms of the equilibrium parameters that we have derived. In the drift formulation, one finds exactly (Magnetic confinement course):

$$\frac{d\psi}{dt} = \frac{m}{Ze} F \frac{d}{dt}\left(\frac{v_\parallel}{B}\right).$$

From Eq. (2.19) we have,

$$\frac{dr}{dt} = \frac{d\psi}{dt}\frac{dr}{d\psi} = \frac{qR_0}{r}\frac{m}{Ze} F \frac{d}{dt}\left(\frac{v_\parallel}{B}\right)$$

Now integrate directly, take $F(r)$ as a constant, noting motion across field lines is slow compared to along field lines (along $\omega(t)$, note e.g. $d\theta/dt \approx v_\parallel/(qR)$), and F depends weakly on r. Hence:

$$r(t) - r(t_0) \approx \frac{q(t_0)R_0}{r(t_0)}\frac{m}{Ze}\left(\frac{v_\parallel[\omega(t)]}{B[\omega(t)]} - \frac{v_\parallel[\omega(t_0)]}{B[\omega(t_0)]}\right)$$

which is an equation for $r(\omega)$. From Eq. (2.28) recall that $v_\parallel$ can be written in terms of $B(\omega)$ and the constants of motion.

For the conventional banana orbit, we choose a 350keV proton, with $\mu/E = 1/B_0 = T^{-1}$ with $r(t_0) = 0.4$ giving trapping angle $\omega_t = \omega(t_0) = \pm 2.26$ rads.

For the tear drop orbits, we choose a 3.5MeV proton, with $\mu/E = 1/B_{\mathrm{min}} - 0.005 = 1.94/B_0 = 1.94T^{-1}$ with $r(t_0) = 0.67$ and giving trapping angles $|\omega_t| = |\omega(t_0)| = 0.54$ rads and $|\omega_t| = |\omega(t_0)| = 0.84$ rads.