Plasma Instabilities

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Content of Plasma Instabilities

1. The Ideal MHD Model and axisymmetric equilibria

2. Grad-Shafranov solutions: toroidal and shaped axisymmetric equilibria

3. Theory of linear ideal MHD stability

4. External kink modes and inertia treatment for ideal and resistive problems

5. Linear and non-linear Tearing Modes

6. Localised toroidal instabilities: ballooning modes and interchange modes

7. Fast ion effects on pressure driven long wavelength instabilities

Lecture 1

The ideal MHD Model and axisymmetric equilibria

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Content of this lecture

1. The Ideal MHD Model and axisymmetric equilibria
   - Kinetic derivation of perpendicular force balance
   - Conservation of flux in ideal MHD
   - Equilibrium force balance
   - Axisymmetric equilibria with cylindrical coordinates
   - Grad-Shafranov equation with flux coordinates

The Ideal MHD Model

- The MHD model is briefly derived here in a way that allows simple adaptation to more realistic (e.g. kinetic - MHD) models. The MHD model assumes that the plasma is collisional, despite the fact that ideal MHD is often applied to the dynamics of almost collisionless tokamak plasmas.
- Nevertheless, ideal MHD does well describe perpendicular (to the equilibrium magnetic field) dynamics of macroscopic instabilities. Cross-magnetic field behaviour is relatively slow. Ideal MHD is not a good descriptor of dynamics parallel to the magnetic field, which for collisionless plasmas, occurs too rapidly for the ideal MHD model.
- Luckily, parallel dynamics are often stabilising, or they introduce other classes of instabilities which can be distinguished experimentally from ideal MHD. For this reason, ideal MHD can be used in order to assess the lowest level (necessary but not sufficient) stability of a magnetised plasma. Despite its limitations, it remains an extremely complex and powerful tool to investigate stability. It is also a closed system, enabling e.g. non-linear effects to be studied in a way that is generally too complicated for any other model.

The Ideal MHD Model

$$
\begin{aligned}
\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \ \text{(Conservation of mass)} \\
\rho \frac{d\boldsymbol{u}}{dt} + \boldsymbol{\nabla} P - \boldsymbol{J} \times \boldsymbol{B} &= \boldsymbol{0} \ \text{(Equation of motion)} \\
\frac{d}{dt}\left(P \rho^{-\gamma}\right) &= 0 \ \text{(Adiabatic equation of state)} \\
\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} &= \boldsymbol{0} \ \text{(Ideal Ohm's law)} \\
\boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \ \text{(Field lines have no sources or sinks)} \\
\boldsymbol{\nabla} \times \boldsymbol{B} - \boldsymbol{J} &= \boldsymbol{0} \ \text{(Ampère's law)} \\
\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{E} &= \boldsymbol{0} \ \text{(Faraday's law),}
\end{aligned}
$$

where $\boldsymbol{u}$ is the fluid velocity, ρ is the mass density, P is the plasma pressure, $\boldsymbol{J}$ is the current density, γ is the adiabaticity index and the convective derivative $d/dt = \partial/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$. The electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ consist of externally applied fields and averaged fields arising from long-range inter-particle interactions.

The Ideal MHD Model

$$
\begin{aligned}
\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 && \text{Exact - no assumptions on gyro radius or collisionality} \\
\rho \frac{d\boldsymbol{u}}{dt} + \boldsymbol{\nabla} P - \boldsymbol{J} \times \boldsymbol{B} &= \mathbf{0} && \text{Anisotropic pressures in collisionless plasmas} \\
\frac{d}{dt}\left(P \rho^{-\gamma}\right) &= 0 && \text{Poor model for system closure in collisionless plasma} \\
\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} &= \mathbf{0} && \text{Reasonable model for small Larmor radius} \\
\boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 && \text{Exact - Maxwell's} \\
\boldsymbol{\nabla} \times \boldsymbol{B} - \boldsymbol{J} &= \mathbf{0} && \text{Pre-Maxwell - good for MHD relevant timescales} \\
\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{E} &= \mathbf{0} && \text{Exact}
\end{aligned}
$$

Luckily, near marginal stability (small growth rates), plasma motions are incompressible. It will be seen that for incompressible motions the parallel momentum equation does not play an important role (though this is only true when there is no plasma rotation - advanced topic outside e.g. Freidberg). The energy equation also has intuitive meaning in the collisionless limit. With this in mind, we derive the MHD model in a non-standard way.

Appropriate Assumptions and Variables

The neglect of $\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}$ (Poisson equation $\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho$, with ρ the charge distribution) implies that QUASI NEUTRALITY is valid everywhere:

$$\sum_j Z_j n_j = n_e \quad \text{with } j \text{ summed over ion species, and } n \text{ is density}$$

It is customary to introduce a mass density ρ rather than number density. Due to quasi-neutrality, and electron/ion mass ratio, fluid mass density in MHD is essentially:

$$m_i n_i + m_e n_e \approx m_i n_i$$

i.e. the momentum of the fluid is carried by the ions. We define the mass density

$$\rho = m_i n_i \tag{1.1}$$

Electrons do make an appearance in the fluid like quantity of the current (difference between ion and electron velocity density):

$$\boldsymbol{J} = e(Z_i n_i \boldsymbol{u}_i - n_e \boldsymbol{u}_e) \tag{1.2}$$

Electrons and ions also contribute to the plasma pressure, $P = P_i + P_e$.

Guiding Centre Motion

The orbit of a single particle, with velocity components $v_\parallel$ and $v_\perp$ and mass m_i (m_e for an electron) and charge e, in an electro-magnetic field can be written as

$$\boldsymbol{v} = \boldsymbol{v}_{\perp \mathrm{gyro}} + v_\parallel \boldsymbol{b} + \boldsymbol{v}_{\perp g}$$

where $\boldsymbol{v}_{\perp \mathrm{gyro}}$ is the rapid gyro motion about a field line, $v_\parallel \boldsymbol{b}$ is the streaming along a field line (basic confinement) and the remainder is the slow guiding centre drift velocity, comprising the sum of $\boldsymbol{E} \times \boldsymbol{B}$, $\boldsymbol{\nabla} B$, curvature $\kappa = (\boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b})$ (where $\boldsymbol{b} = \boldsymbol{B}/B$), and polarisation drifts for ions and electrons:

$$\boldsymbol{v}_{\perp gi} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} + \frac{\boldsymbol{b}}{\Omega_{ci}} \times \left[\frac{v_\perp^2}{2} \frac{\boldsymbol{\nabla} B}{B} + v_\parallel^2 \kappa + \frac{d}{dt} \left(\frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} \right) \right] \tag{1.3}$$

$$\boldsymbol{v}_{\perp ge} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} - \frac{\boldsymbol{b}}{\Omega_{ce}} \times \left[\frac{v_\perp^2}{2} \frac{\boldsymbol{\nabla} B}{B} + v_\parallel^2 \kappa + \frac{d}{dt} \left(\frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} \right) \right] \tag{1.4}$$

where $\Omega_{e,i} = eZ_{e,i}B/m_{e,i}$ ($Z_e = -1$) etc. So, some important properties to note: the $\boldsymbol{E} \times \boldsymbol{B}$ drift is in the same direction for both ions and electrons, and forms the basis for the MHD fluid velocity. Corrections are in the opposite direction for electrons and ions, so thus, create the MHD current. Corrections are r_L/a smaller, where $r_L = v_\perp/\Omega_c$.

Take the distribution function of ions F_i and F_e (recall that $\int dv^3 F_{i,e} = n_{i,e}$) and evaluate the first moment of the distribution functions in order to obtain the 'fluid' velocity to leading order in gyro radius (this neglects Hall terms and resulting diamagnetic effects):

$$\boldsymbol{u}_{\perp i,e} = \frac{1}{n_{i,e}} \int dv^3 F_{i,e} \boldsymbol{v}_{\perp gi,e} \approx \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}, \tag{1.5}$$

which can be re-written as Ohm's law, which states that the electric field in a frame moving with the plasma (fluid velocity $\boldsymbol{u}$ is zero:

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = 0$$

Conservation of Flux in Ideal MHD

Consider a closed loop C of surface S drawn in the fluid. A magnetic field $\boldsymbol{B}$ passes through the loop, bounded by l, and so the flux Φ linking the loop is

$$\Phi = \int_S \boldsymbol{B} \cdot \boldsymbol{dS}$$

The flux can change in two ways.
1) The loop can deform, and by altering its shape, it can lose or capture field lines. If the loop moves with velocity $\boldsymbol{v}_c$, and $\boldsymbol{\delta l}$ is an elemental length tangential to the loop L, then in time δt it will sweep out an elemental area:

$$\boldsymbol{\delta S}_c = \boldsymbol{v}_c\, \delta t \times \boldsymbol{\delta l} = \boldsymbol{v}_c\, \delta t\, \delta l\ \sin\Theta$$

and the flux linking this element is

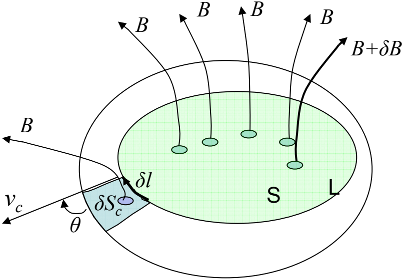
$$\delta\Phi_c = (\boldsymbol{v}_c\, \delta t \times \boldsymbol{\delta l}) \cdot \boldsymbol{B} = -(\boldsymbol{v}_c \times \boldsymbol{B}) \cdot \boldsymbol{\delta l}\, \delta t.$$

2) Alternatively, the flux can change through $\boldsymbol{B}$ itself changing in time. The change in flux through a stationary element of area $\boldsymbol{\delta S}$ in time δt is

$$\delta\Phi_t = \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{\delta S}\, \delta t$$

and the total rate of change of flux is therefore the sum of 1) and 2):

$$\frac{\delta\boldsymbol{\Phi}}{\delta t} = \frac{\delta\boldsymbol{\Phi}_c}{\delta t} + \frac{\delta\boldsymbol{\Phi}_t}{\delta t}.$$



Concept of Conservation of Flux in Ideal MHD

On proceeding to the limit where δt, $\boldsymbol{\delta S}$ and $\boldsymbol{\delta l}$ tend to zero, and summing the contributions from the elemental loops, it follows that,

$$\frac{d\boldsymbol{\Phi}}{dt} = \oint_L (\boldsymbol{v}_c \times \boldsymbol{B}) \cdot \boldsymbol{dl} + \int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{dS},$$

where the line integral evaluated around the loop L. Use Faraday's law $\partial \boldsymbol{B}/\partial t = -\boldsymbol{\nabla} \times \boldsymbol{E}$, and employ Stoke's theorem $\int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \boldsymbol{dS} = \oint_L \boldsymbol{A} \cdot \boldsymbol{dl}$ to obtain,

$$\frac{d\boldsymbol{\Phi}}{dt} = \oint_L (\boldsymbol{v}_c \times \boldsymbol{B}) \cdot \boldsymbol{dl} - \oint_L \boldsymbol{E} \cdot \boldsymbol{dl}$$

Apply now resistive Ohm's law $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - \eta \boldsymbol{J} = 0$, with η the resistivity and $\boldsymbol{v}$ the *fluid* velocity, giving:

$$\frac{d\boldsymbol{\Phi}}{dt} = -\oint_L \{(\boldsymbol{v}_c - v) \times \boldsymbol{B} + \eta \boldsymbol{J}\} \cdot \boldsymbol{dl}$$

If we define the boundary $\boldsymbol{S}$, and the increment $\boldsymbol{\delta S}$ as those of the fluid, such that the fluid moves with velocity $\boldsymbol{v}_c$, then

$$\frac{d\boldsymbol{\Phi}}{dt} = -\oint_L \eta \boldsymbol{J} \cdot \boldsymbol{dl}.$$

Concept of Conservation of Flux in Ideal MHD

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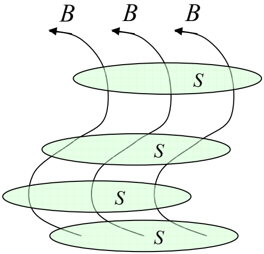


If we define the boundary $\boldsymbol{S}$, and the increment $\boldsymbol{\delta S}$ as those of the fluid, such that the fluid moves with velocity $\boldsymbol{v}_c$, then

$$\frac{d\boldsymbol{\Phi}}{dt} = -\oint_L \eta \boldsymbol{J} \cdot \boldsymbol{dl}.$$

Plainly in the ideal limit we have the conceptually appealing notion of field lines being frozen into the fluid. For, if the fluid elements retain their identity it follows that the magnetic field topology is necessarily invariant. This constraint has a profound effect on the allowable class of MHD motions. Clearly, a small amount of resistivity would allow the field lines to diffuse through the fluid (e.g. sun spots and solar flare activity, as well as tearing modes in tokamaks).

Current and Momentum Equation

Evaluate the guiding centre current by subtracting Eqs. (1.3) and (1.4) and evaluating the first moments of the electron and ion distributions

$$\boldsymbol{J}_{\perp g} = e\int dv^3 (F_i \boldsymbol{v}_{\perp gi} - F_e \boldsymbol{v}_{\perp ge}) = \frac{\boldsymbol{b}}{B} \times \left[P_\perp \frac{\boldsymbol{\nabla} B}{B} + P_\parallel \kappa + \rho \frac{d}{dt}\left(\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\right)\right] \qquad (1.6)$$

where

$$P_\perp = P_{\perp i} + P_{\perp e} = m_i \int dv^3 F_i \frac{v_\perp^2}{2} + m_e \int dv^3 F_e \frac{v_\perp^2}{2}$$

$$P_\parallel = P_{\parallel i} + P_{\parallel e} = m_i \int dv^3 F_i v_\parallel^2 + m_e \int dv^3 F_e v_\parallel^2$$

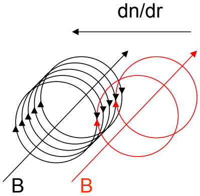
$$\rho \approx m_i \int dv^3 F_i$$

We also need to consider the magnetisation (diamagnetic) current

$$\boldsymbol{J}_M = -\boldsymbol{\nabla} \times \left(P_\perp \frac{\boldsymbol{B}}{B^2}\right)$$

The total current is the sum $\boldsymbol{J} = \boldsymbol{J}_{\perp g} + \boldsymbol{J}_M$. The trick is so substitute Ohm's law (1.5) into the guiding centre current (1.6) and form the cross product with $\boldsymbol{B}$ in order to obtain the momentum equation:

$$\rho\left(\boldsymbol{b} \times \frac{du_\perp}{dt}\right) \times \boldsymbol{b} = \boldsymbol{J} \times \boldsymbol{B} - \boldsymbol{\nabla}_\perp \cdot \underline{\underline{P}}$$



If there is a density gradient, a current is generated perpendicular to B and dn/dr. Larmor radius (and currents) also affected by temperature (v^2) and field strength.

Momentum Equation continued

Here we have

$$\underline{\underline{P}} = P_\perp \underline{\underline{I}} + (P_\parallel - P_\perp)\boldsymbol{b}\boldsymbol{b} = \begin{pmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{pmatrix} \quad \text{and} \quad \boldsymbol{\nabla}_\perp = \boldsymbol{\nabla} - \boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla}) \tag{1.7}$$

where $\underline{\underline{I}}$ is the unit dyadic. This diagonal pressure tensor, and properties of curvature vector reveal the useful result:

$$(\boldsymbol{\nabla}\cdot\underline{\underline{P}}_\perp = [\boldsymbol{\nabla} - \boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla})]P_\perp + (P_\parallel - P_\perp)\boldsymbol{\kappa}$$

Let us recap the model equations so far:

$$\begin{aligned} \frac{d\rho}{dt} + \rho\boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 \\ \rho\left.\frac{d\boldsymbol{u}_\perp}{dt}\right|_\perp - \boldsymbol{J}\times\boldsymbol{B} + \boldsymbol{\nabla}_\perp\cdot\underline{\underline{P}} &= \boldsymbol{0} \\ \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} &= \boldsymbol{0} \\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0 \\ \boldsymbol{\nabla}\times\boldsymbol{B} - \boldsymbol{J} &= \boldsymbol{0} \\ \frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{\nabla}\times\boldsymbol{E} &= \boldsymbol{0} \end{aligned} \tag{1.8}$$

We note that we need to close the system with an energy equation (giving $\underline{\underline{P}}$), and an equation for $u_\parallel$. Depending on the application, we will close the system as appropriate.

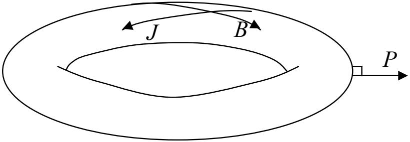
MHD Equilibrium

We initially look at a static equilibrium, meaning that there are no equilibrium flows $\boldsymbol{u} = 0$ and $\partial \boldsymbol{u}/dt = 0$. Consider the isotropic case where $P_\perp = P_\parallel = P$. Thus we have to solve

$$\boldsymbol{J} \times \boldsymbol{B} = \boldsymbol{\nabla} P \tag{1.9}$$

On taking the divergence of Ampère's law $\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{J}$ it follows that $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$, indicating that there are no sources or sinks in the current, in accordance with charge neutrality. It is evident from Eq.(1.9) that $\boldsymbol{J} \cdot \boldsymbol{\nabla} P = 0$ and $\boldsymbol{B} \cdot \boldsymbol{\nabla} P = 0$ which means that lines of magnetic field and current lie on surfaces of constant pressure. In a tokamak these isobaric surfaces are known as flux surfaces

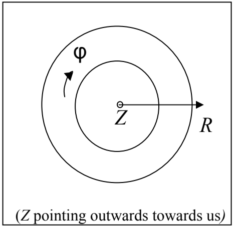
- The so called 'hairy ball' theorem, attributed to Poincaré, states that the only smooth surface which can be covered by a non-vanishing vector field (i.e. current of magnetic field vectors) is a toroidal one.
- Hence toroidal surfaces of constant pressure (flux surfaces) will have the attractive property that the fields of $\boldsymbol{B}$ and $\boldsymbol{J}$ be non-zero everywhere on the surface.
- This indicates that the $\boldsymbol{J} \times \boldsymbol{B}$ force can, in principle at least, balance the pressure everywhere.



Definitions of Cylindrical Coordinates

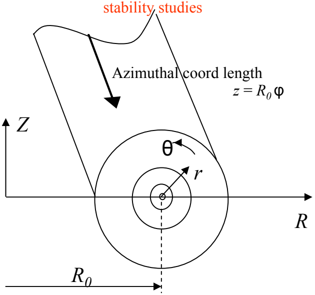
We note that there are two types of cylindrical coordinates used in our field. The first type represents a cylindrical system which is an un-approximated definition of the toroidal plasma. The other type is used in stability studies, and involves approximating to an infinite aspect ratio ($R/a \to \infty$), and circular nested flux surfaces. We should not confuse these very different meanings of 'cylindrical.' Both will be used in this course.

Cylindrical coordinates of un-approximated torus looking down from above



(R, Z, φ)

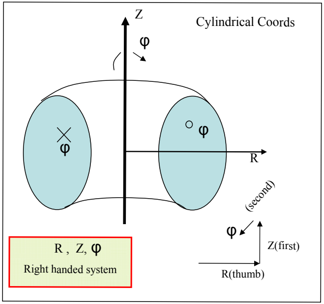
Approximated cylindrical tokamak used in stability studies

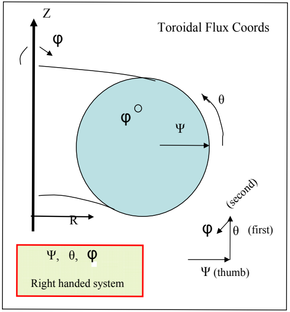


(r, θ, z)

Right Handed Cylindrical and Flux Coordinates

We use an unconventional ordering (R, Z, ϕ) for cylindrical coordinates (usual ordering is (R, ϕ, Z). The unconventional ordering allows a smooth and easy transformation to flux coordinates (ψ, Θ, ϕ), where ϕ is again the third coordinate. Much confusion could be avoided in plasma physics if everyone started from cylindrical (R, Z, ϕ). See Goedbloed and appendices for more information, and vector calculus identities. Many of you will have already come across the consequences of this problem





Axisymmetric Toroidal Equilbria

We employ the cylindrical coordinate system (R, Z, ϕ), where (see notes pages):

$$\boldsymbol{B} = B_R \boldsymbol{e}_R + B_Z \boldsymbol{e}_Z + B_\phi \boldsymbol{e}_\phi, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{1}{R}\frac{\partial (R B_R)}{\partial R} + \frac{\partial B_Z}{\partial Z} + \frac{1}{R}\frac{\partial B_\phi}{\partial \phi} = 0$$

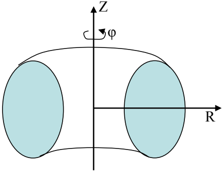
with $\boldsymbol{e}_{R,Z,\phi}$ orthogonal **unit** vectors. We assume equilibrium axisymmetry, which means that all equilibrium quantities (e.g. pressure magnetic field and current) are independent of ϕ, so that $B = B(Z, R)$. This then allows us to introduce a stream function to represent the field components in the poloidal plane

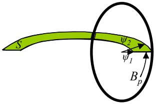
$$R B_R = -\frac{\partial \psi}{\partial Z} \quad and \quad R B_Z = \frac{\partial \psi}{\partial R}$$

or more succinctly,

$$\boldsymbol{B} = B_\phi \boldsymbol{e}_\phi + \boldsymbol{B}_p \ \text{ with } \ \boldsymbol{B}_p = \boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi \ \text{ and } \ \boldsymbol{\nabla}\phi = \frac{\boldsymbol{e}_\phi}{R} \tag{1.10}$$

It is clear that toroidal and poloidal fields are perpendicular to $\boldsymbol{\nabla}\psi$, i.e. the field lines lie on surfaces of constant ψ. It is now clear why these surfaces are named flux surfaces, since ψ is related to the poloidal flux ψ_p, for instance through a ring in the equitorial plane defined by $S = \{Z = 0, R(\psi_1) < R < R(\psi_2)\}$





$$\begin{aligned} \psi_p &= \int_S \boldsymbol{B}_p \cdot d\vec{S} = \int_S \boldsymbol{\nabla} \times (\psi \boldsymbol{\nabla}\phi) \cdot \boldsymbol{dS} \\ &= \oint \psi \boldsymbol{\nabla}\phi \cdot \boldsymbol{dl} = 2\pi(\psi_2 - \psi_1) \ (1.11) \end{aligned}$$

The total poloidal flux (vacuum as well as plasma field) through the circular magnetic axis is found by taking $R(\psi_1) = R_0$ and $R(\psi_2) = 0$, where $R = R_0, Z = 0$ defines the magnetic axis.

Axisymmetric Toroidal Equilibria

Applying Ampère's law in the cylindrical system we obtain (see exercises)

$$J_R = \frac{\partial B_\phi}{\partial Z} \quad , \quad J_Z = -\frac{1}{R}\frac{\partial (RB_\phi)}{\partial R} \quad , \quad J_\phi = \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) + \frac{1}{R}\frac{\partial^2 \psi}{\partial Z^2}.$$

Grouping the poloidal and toroidal currents, identified in Eq. (1.10), one obtains the compact expression

$$\boldsymbol{J} = \frac{1}{R}\boldsymbol{\nabla}(RB_\phi) \times \boldsymbol{e}_\phi + \frac{1}{R}\Delta^* \psi \boldsymbol{e}_\phi \tag{1.12}$$

with Δ^* the Laplacian-like Grad-Shafranov operator, which in general coordinates is given by

$$\Delta^* = R^2 \boldsymbol{\nabla} \cdot \left(\frac{1}{R^2}\boldsymbol{\nabla}\right) \tag{1.13}$$

while in the cylindrical coordinate system, we have

$$\Delta^* = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{\partial^2}{\partial Z^2}.$$

Now, substituting the field Eq. (1.10) and the current Eq. (1.12) into force balance $\boldsymbol{J} \times \boldsymbol{B} - \boldsymbol{\nabla} P = 0$, and forming the dot product with $\boldsymbol{e}_\phi$, (for which $\boldsymbol{\nabla} P \cdot \boldsymbol{e}_\phi = 0$) yields an equation for the toroidal field that is independent of pressure,

$$\frac{\partial (RB_\phi)}{\partial R}\frac{\partial \psi}{\partial Z} - \frac{\partial (RB_\phi)}{\partial Z}\frac{\partial \psi}{\partial R} = 0.$$

This states that the Jacobian of the functions RB_ϕ and ψ is zero. The vanishing of the Jacobian implies that

$$F(\psi) = RB_\phi \quad \text{is constant on a flux surface} \tag{1.14}$$

Axisymmetric Toroidal Equilibria

Now use the fact that field lines lie on isobaric surfaces $\boldsymbol{B} \cdot \boldsymbol{\nabla} P = 0$:

$$-\boldsymbol{B} \cdot \boldsymbol{\nabla} P = \frac{1}{R} \boldsymbol{\nabla} \psi \times \boldsymbol{e}_\phi \cdot \boldsymbol{\nabla} P = \frac{1}{R} \frac{\partial \psi}{\partial R} \frac{\partial P}{\partial Z} - \frac{1}{R} \frac{\partial \psi}{\partial Z} \frac{\partial P}{\partial R} = 0$$

which is the Jacobian of P and ψ, which again vanishes, so that $P = P(\psi)$. Finally, forming the dot product of $\boldsymbol{\nabla} P = \boldsymbol{J} \times \boldsymbol{B}$, with $\boldsymbol{e}_R$ we obtain

$$\frac{\partial P}{\partial R} = -\left(\frac{1}{R} \frac{\partial (R B_\phi)}{\partial R} \right) B_\phi - \frac{1}{R} \Delta^* \psi \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right).$$

Using the fact that both P and $R B_\phi$ are functions of ψ only:

$$\frac{\partial P}{\partial R} = \frac{\partial \psi}{\partial R} \frac{dP}{d\psi} = -\frac{\partial \psi}{\partial R} \frac{1}{R^2} F \frac{dF}{d\psi} - \frac{\partial \psi}{\partial R} \frac{\Delta^* \psi}{R^2},$$

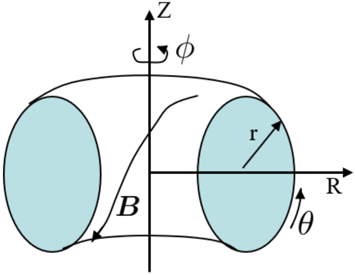
which directly gives the Grad-Shafranov Equation

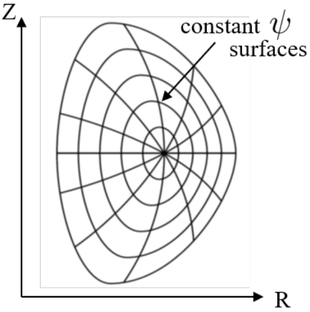
$$\Delta^* \psi = -R^2 \frac{dP(\psi)}{d\psi} - F(\psi) \frac{dF(\psi)}{d\psi}. \tag{1.15}$$

Note that on substitution of Eq.(1.13) into Eq.(1.15) we see the true power of the Grad-Shafranov equation. In particular, it is independent of the system of coordinates on a given flux surface.

Equation (1.15) is a non-linear, second order elliptic partial differential equation for the equilibrium in terms of the flux. The general procedure for finding its solution is first to prescribe the pressure $P(\psi)$ and current function $F(\psi)$ as some physically reasonable distribution of the flux ψ.

Flux Coordinates for Equilibrium problems





- The equilibrium problem in axisymmetry requires solving for the locations of constant ψ surfaces over the R, Z plane.
- For flux coordinate problem, we define a minor radius coordinate r for which ψ is a constant on constant r surfaces. For mapping these surfaces into R, Z we require also a poloidal coordinate Θ. The complete coordinate system is (r, Θ, ϕ)
- This flux coordinate system is not unique. We choose the most convenient, depending on the problem investigated.

Flux Coordinates

Of great importance in equilibrium calculations, but also in the modelling of plasma transport and in stability analysis, are flux coordinates (r, Θ, ϕ). Here ϕ is the usual toroidal angle. The radial coordinate $r(\psi)$ labels the flux surfaces. It can be the poloidal (or toroidal) flux itself, or the volume enclosed by each flux surface, or can be chosen to closely resemble the minor radius (distance to the magnetic axis). The various definitions used for the poloidal angle Θ, however, are convenient in very specific applications: (i) the proper geometrical angle can be used when the geometry is fixed, for instance in tomographic diagnostic methods, (ii) an orthogonal coordinate system ($\boldsymbol{\nabla} r \cdot \boldsymbol{\nabla}\Theta = 0$) can be convenient in ballooning stability analysis and certain means of solving the Grad-Shafranov equation, (iii) and most universally applied, especially in stability studies, are coordinates in which the field lines appear straight. In these coordinates the local pitch of the magnetic field line trajectory

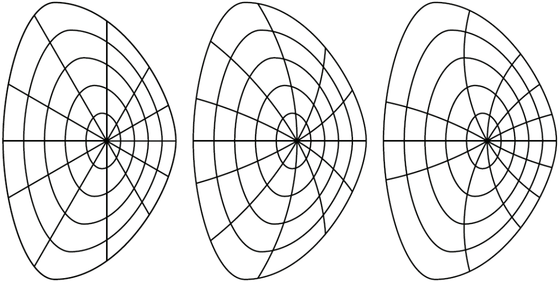
$$q_l = \frac{d\phi}{d\Theta} \tag{1.16}$$

is a constant on each flux surface. Note that the standard definition of the safety factor q if the average of q_l over the poloidal angle,

$$q(\psi) = \frac{1}{2\pi}\int_0^{2\pi} q_l(\Theta, \psi)\, d\Theta \tag{1.17}$$

Hence q is invariant on the flux surface, i.e. $q = q(\psi)$. For straight field line coordinates, $q_l = q$, so that $d\phi/d\Theta$ is also a flux surface quantity (by definition). There are other straight field line coordinates such as Boozer coordinates. Here, the toroidal angle and poloidal angles are modified in order to permit the field to be written in a simplified way in covariant form, and allow straight field lines even when the equilibrium is not axisymmetric (e.g. stellarator). A problem with straight field line coordinates is poor resolution on the low field side (LFS), especially for low aspect ratio, and high pressure (e.g. for modelling spherical tokamaks). The equilibrium will be expanded analytically with flux coordinates not fulfilling (i), (ii), or (iii).

Types of flux coordinates



(i) Proper geometric angle (ii) Straight field line system (iv) Orthogonal system

Flux Coordinates

Recall that the Jacobian $\mathcal{J}$ of any coordinate system (r, Θ, ϕ) is such that the volume element is given by

$$dv^3 = dr\, d\Theta\, d\phi\, \mathcal{J}$$

For general non-orthogonal flux coordinates, the Jacobian is written in its general form

$$\mathcal{J} = (\boldsymbol{\nabla} r \cdot \boldsymbol{\nabla}\Theta \times \boldsymbol{\nabla}\phi)^{-1}\,. \tag{1.18}$$

Meanwhile, the square of element of length dl along the magnetic field comprises the following sum over the components of the metric tensor g_{ij}, with $\mathcal{J} = \sqrt{\det(g_{i,j})}$:

$$dl^2 = g_{r,r}dr^2 + 2g_{r,\Theta}dr\, d\Theta + g_{\Theta,\Theta}d\Theta^2 + g_{\phi,\phi}d\phi^2 \tag{1.19}$$

with

$$g_{r,r} = \frac{\mathcal{J}^2}{R^2}|\boldsymbol{\nabla}\Theta|^2, \;\; g_{r,\Theta} = -\frac{\mathcal{J}^2}{R^2}\boldsymbol{\nabla}\Theta\cdot\boldsymbol{\nabla} r, \;\; g_{\Theta,\Theta} = \frac{\mathcal{J}^2}{R^2}|\boldsymbol{\nabla} r|^2 \;\text{ and }\; g_{\phi,\phi} = R^2. \tag{1.20}$$

We now employ these definitions in the Grad-Shafranov equation Eq.(1.15) (with Eq.(1.13)). Useful are the following operators on a scalar f and vector $\boldsymbol{Y}$:

$$\boldsymbol{\nabla} f = \boldsymbol{\nabla} r\frac{\partial f}{\partial r} + \boldsymbol{\nabla}\Theta\frac{\partial f}{\partial\Theta} + \boldsymbol{\nabla}\phi\frac{\partial f}{\partial\phi}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{Y} = \frac{1}{\mathcal{J}}\left\{\frac{\partial}{\partial r}(\mathcal{J}\boldsymbol{\nabla} r\cdot\boldsymbol{Y}) + \frac{\partial}{\partial\Theta}(\mathcal{J}\boldsymbol{\nabla}\Theta\cdot\boldsymbol{Y}) + \frac{\partial}{\partial\phi}(\mathcal{J}\boldsymbol{\nabla}\phi\cdot\boldsymbol{Y})\right\} \tag{1.21}$$

The crucial point is that the Grad-Shafranov equation is in terms of a divergence operator, see Eq.(1.13), which is straightforward to obtain for non-orthogonal coordinates (see notes pages and exercise series). We obtain:

$$\Delta^*\psi = \frac{R^2}{\mathcal{J}}\left[\frac{\partial}{\partial r}\left(\frac{\psi' g_{\Theta,\Theta}}{\mathcal{J}}\right) - \frac{\partial}{\partial\Theta}\left(\frac{\psi' g_{r,\Theta}}{\mathcal{J}}\right)\right] \tag{1.22}$$

where

$$\boldsymbol{\nabla}\psi(r) = \psi'\boldsymbol{\nabla} r, \quad \textit{where} \;\; X' = \frac{dX}{dr} \tag{1.23}$$

Notes

Derivation of Grad-Shafranov Equation in Flux Coordinates:
Begin with the general divergence relation of Eq. (1.21),

$$\boldsymbol{\nabla}\cdot\boldsymbol{Y}=\frac{1}{\mathcal{J}}\left\{\frac{\partial}{\partial r}(\mathcal{J}\boldsymbol{\nabla}r\cdot\boldsymbol{Y})+\frac{\partial}{\partial\Theta}(\mathcal{J}\boldsymbol{\nabla}\Theta\cdot\boldsymbol{Y})+\frac{\partial}{\partial\phi}(\mathcal{J}\boldsymbol{\nabla}\phi\cdot\boldsymbol{Y})\right\},$$

Now, we wish to obtain

$$\Delta^{*}\psi=R^2\boldsymbol{\nabla}\cdot\left(\frac{1}{R^2}\boldsymbol{\nabla}\right)\psi=R^2\boldsymbol{\nabla}\cdot\left(\frac{1}{R^2}\boldsymbol{\nabla}\psi\right),$$

so that in the above substitute,

$$\boldsymbol{Y}=\left(\frac{1}{R^2}\boldsymbol{\nabla}\psi\right)$$

giving

$$\boldsymbol{\nabla}\cdot\boldsymbol{Y}=\boldsymbol{\nabla}\cdot\left(\frac{1}{R^2}\boldsymbol{\nabla}\psi\right)=\frac{1}{\mathcal{J}}\left\{\frac{\partial}{\partial r}\left(\frac{\mathcal{J}}{R^2}\boldsymbol{\nabla}r\cdot\boldsymbol{\nabla}\psi\right)+\frac{\partial}{\partial\Theta}\left(\frac{\mathcal{J}}{R^2}\boldsymbol{\nabla}\Theta\cdot\boldsymbol{\nabla}\psi\right)+\frac{\partial}{\partial\phi}\left(\frac{\mathcal{J}}{R^2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi\right)\right\}$$

Now, the poloidal plane is perpendicular to the ϕ direction, so that $\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi=0$. Using also Eq. (1.23), $\boldsymbol{\nabla}\psi(r)=\psi'\boldsymbol{\nabla}r$ where $X'=dX/dr$, and Eqs. (1.20), i.e. ($g_{r,\Theta}=-(\mathcal{J}^2/R^2)\boldsymbol{\nabla}\Theta\cdot\boldsymbol{\nabla}r$ and $g_{\Theta,\Theta}=(\mathcal{J}^2/R^2)|\boldsymbol{\nabla}r|^2$) easily gives Eq. (1.22)

$$\Delta^{*}\psi=\frac{R^2}{\mathcal{J}}\left[\frac{\partial}{\partial r}\left(\frac{\psi' g_{\Theta,\Theta}}{\mathcal{J}}\right)-\psi'\frac{\partial}{\partial\Theta}\left(\frac{g_{r,\Theta}}{\mathcal{J}}\right)\right]$$

Aspect ratio expanded equilibria

For making analytic progress we will perform a Fourier expansion in R and Z :

$$R(r,\Theta) = R_0 + r\cos\Theta - \Delta(r) + \sum_{m=2}^{\infty} S_m(r)\cos(m-1)\Theta + P(r)\cos\Theta \quad (1.24)$$

$$Z(r,\Theta) = r\sin\Theta - \sum_{m=2}^{\infty} S_m(r)\sin(m-1)\Theta + P(r)\sin\Theta, \quad (1.25)$$

where R_0 is the major radius at the magnetic axis, r is a flux label, and Δ is the Shafranov shift (this expansion is essentially the axisymmetric limit of the VMEC code). Here $P(r)$ are small corrections (as seen later).

Comparing this with another standard way of writing the equilibrium (e.g. as described in special CHEASE ad-hoc equilibrium),

$$R = \hat{R}_0 + \hat{r}\cos(\omega + \delta\sin\omega) \quad (1.26)$$

$$Z = \hat{r}\kappa\sin\omega, \quad (1.27)$$

where the squareness and higher harmonics are ignored, and $\hat{R}_0 = [R(\Theta=0) + R(\Theta=\pi)]/2$ and $\hat{r} = [R(\Theta=0) - R(\Theta=\pi)]/2$ are respectively the major and minor geometric radii. One can identify this equilibrium in terms of the parameters in Eqs. (1.24) and (1.25) with

$$\hat{R_0} = R_0 - \Delta + S_3, \quad \hat{r} = r + S_2.$$

Also, upon choosing $R(\Theta=\pi) = R(\omega=\pi)$ and $Z(\Theta=\pi) = Z(\omega=\pi)$ we have

$$\kappa = \frac{r - S_2}{r + S_2}. \quad (1.28)$$

Thus the $S_2(r)$ profile defines the elongation profile. Also, a best fit between these coordinates gives,

$$\delta = \frac{4S_3}{r},$$

where $S_3(r)$ defines the triangularity profile).

Boundary conditions for equilibria

Realistic boundary conditions (BC) are obviously required, and these come in essentially two forms.

1. A sufficient BC would be to define $\psi(R, Z)$ on a closed contour, i.e. by defining the shape of one flux surface, for instance. If a fixed outer surface is specified, then in essence the plasma-vacuum boundary is replaced by the surface of a perfect conductor (on which ψ is necessarily constant). This is a fixed boundary condition, which defines ψ in the entire plasma.
2. By specifying a flux surface in the vacuum region, one has a free boundary problem. Taking into account the currents in the coils leads to a somewhat different approach. One can use the known currents in the coils and an assumed plasma current distribution to compute ψ on a boundary which is convenient for the computations, a rectangle, say. With these Dirichlet boundary conditions one then solves the Grad-Shafranov equation in the interior. This leads to a different plasma current distribution than originally assumed, and one iterates the procedure. As an alternative, or in addition to considering the coil currents and computing $\boldsymbol{B}$ in the metal parts as well as the vacuum, poloidal field measurements can be available close to the coils or near the plasma. This makes the set of boundary conditions altogether more inhomogeneous and a very adaptable equilibrium solver is required. The system can even be over-determined.

Notes

Contravariant and Basis Vectors:
Define the cartesian basis

$$\vec{c} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$

where $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ are contravariant basis vectors. For a cartesian system, the covariant basis vectors are simply

$$\vec{e}^x = \frac{\partial \vec{c}}{\partial x} = \vec{e}_x, \quad \vec{e}^y = \frac{\partial \vec{c}}{\partial y} = \vec{e}_y, \quad \vec{e}^z = \frac{\partial \vec{c}}{\partial z} = \vec{e}_z$$

We now define a set of general coordinates $[u^1, u^2, u^3]$ (e.g. these might be $[R, Z, \phi]$ or $[r, \Theta, \phi]$). These coordinates have associated basis:

$$\text{Contravariant basis :} \qquad \boldsymbol{e}_i = \frac{\partial \vec{c}}{\partial u^i}, \quad i = 1, 2, 3$$

$$\text{Covariant basis :} \qquad \boldsymbol{e}^i = \vec{\nabla} u^i, \quad i = 1, 2, 3$$

These are related to each other as follows, through the Jacobian $\mathcal{J}$

$$\vec{e}_i = \mathcal{J}\vec{e}^j \times \vec{e}^k, \qquad \mathcal{J} = \frac{1}{\vec{e}^i \cdot (\vec{e}^j \times \vec{e}^k)}$$

$$\vec{e}^i = \mathcal{J}^{-1}\vec{e}_j \times \vec{e}_k, \qquad \mathcal{J} = \vec{e}_i \cdot (\vec{e}_j \times \vec{e}_k)$$

e.g. where $i = 1$, $j = 2$ and $k = 3$.
Any vector $\boldsymbol{B}$ can then be written as follows

$$\text{Contravariant form :} \qquad \vec{B} = B^1\vec{e}_1 + B^2\vec{e}_2 + B^3\vec{e}_3,$$

$$\text{Covariant form :} \qquad \vec{B} = B_1\vec{e}^1 + B_2\vec{e}^2 + B_3\vec{e}^3,$$

where B^i are the contravarient vector components, and B_i are the covariant vector components. The following identify holds for any basis, $\vec{e}^i\boldsymbol{e}_j = 1$ for $i \neq j$, and $\vec{e}^i\boldsymbol{e}_j = 0$ for $i = j$, so that the contravariant vector components can be written in terms of the projection of the vector $\vec{B}$ and the covariant basis vector components:

$$B^i = \vec{B} \cdot \vec{e}^i = \vec{B} \cdot \vec{\nabla}\vec{u}^i, \quad i = 1, 2, 3$$

Notes

Metric tensors and Jacobians:

The contravariant matrix tensor is defined as

$$g_{i,j} = \begin{pmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 & \vec{e}_1 \cdot \vec{e}_3 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 & \vec{e}_2 \cdot \vec{e}_3 \\ \vec{e}_3 \cdot \vec{e}_1 & \vec{e}_3 \cdot \vec{e}_2 & \vec{e}_3 \cdot \vec{e}_3 \end{pmatrix}$$

where off diagonal components are non-zero for a non-orthogonal system. The Jacobian is

$$\mathcal{J} = \sqrt{|g_{i,j}|} = \vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)$$

Note that $\mathcal{J} > 0$ for a right handed coordinate system (we will always adopt right handed coordinate systems in this course).

The covariant matrix tensor is defined as

$$g^{i,j} = \begin{pmatrix} \vec{e}^1 \cdot \vec{e}^1 & \vec{e}^1 \cdot \vec{e}^2 & \vec{e}^1 \cdot \vec{e}^3 \\ \vec{e}^2 \cdot \vec{e}^1 & \vec{e}^2 \cdot \vec{e}^2 & \vec{e}^2 \cdot \vec{e}^3 \\ \vec{e}^3 \cdot \vec{e}^1 & \vec{e}^3 \cdot \vec{e}^2 & \vec{e}^3 \cdot \vec{e}^3 \end{pmatrix}$$

where off diagonal components are non-zero for a non-orthogonal system. The Jacobian is

$$\mathcal{J} = \frac{1}{\sqrt{|g^{i,j}|}} = \frac{1}{\vec{e}^1 \cdot (\vec{e}^2 \times \vec{e}^3)}.$$

Finally note the important relations:

$$B_i = g_{i,j} B^j, \quad B^i = g^{i,j} B_j.$$

Notes

Vector Calculus (we need only grad and div here):

Consider the scalar f we have,

$$\vec{\nabla} f = \vec{e}^1 \frac{\partial f}{\partial u^1} + \vec{e}^2 \frac{\partial f}{\partial u^2} + \vec{e}^3 \frac{\partial f}{\partial u^3} = \sum_i \vec{e}^i \frac{\partial f}{\partial u^i} .$$

Consider now the divergence of the vector $\vec{B}$ in contravariant form

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= \vec{\nabla} \cdot (B^1 \vec{e}_1 + B^2 \vec{e}_2 + B^3 \vec{e}_3) \\ &= \vec{\nabla} \cdot (B^i \vec{e}_i) \end{aligned}$$

using the identities given earlier,

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= \vec{\nabla} \cdot (B^i \vec{e}_i) \\ &= \vec{\nabla} \cdot (B^i \mathcal{J} \vec{e}^j \times \vec{e}^k) \\ &= (\vec{e}^j \times \vec{e}^k) \cdot \vec{\nabla}(B^i \mathcal{J}) + B^i \mathcal{J} \cancelto{0}{\vec{\nabla} \cdot (\vec{\nabla} u^j \times \vec{\nabla} u^k)} \\ &= \mathcal{J}^{-1} \vec{e}_i \cdot \vec{\nabla}(B^i \mathcal{J}) \\ &= \sum_i \mathcal{J}^{-1} \vec{e}_i \cdot \vec{e}^i \frac{\partial (B^i \mathcal{J})}{\partial u^i} \\ &= \sum_i \mathcal{J}^{-1} \frac{\partial (B^i \mathcal{J})}{\partial u^i} \\ &= \mathcal{J}^{-1} \left(\frac{\partial (\mathcal{J} B^1)}{\partial u^1} + \frac{\partial (\mathcal{J} B^2)}{\partial u^2} + \frac{\partial (\mathcal{J} B^3)}{\partial u^3} \right) \end{aligned}$$

Notes

(R, Z, ϕ) coordinate system:

Notice the order of this coordinate system is different from the standard cylindrical system, reason being that we will wish later in the course to convert to flux coordinates (r, Θ, ϕ) with ϕ appearing at the same order. In terms of cartesian system (x, y, z):

$$\begin{aligned} x &= R\cos\phi \\ y &= -R\sin\phi \\ z &= Z \end{aligned}$$

where $\vec{c} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ and $x^2 + y^2 = R^2$.

Contravariant identities:

$$\begin{aligned} \vec{e}_R &= \partial\vec{c}/\partial R = \cos\phi\,\vec{e}_x - \sin\phi\,\vec{e}_y & |\vec{e}_R| &= 1 \\ \vec{e}_Z &= \partial\vec{c}/\partial Z = \vec{e}_z & |\vec{e}_Z| &= 1 \\ \vec{e}_\phi &= \partial\vec{c}/\partial\phi = -R\sin\phi\,\vec{e}_x - R\cos\phi\,\vec{e}_y & |\vec{e}_\phi| &= R \end{aligned}$$

Notice from inspection the important identities:

$$\begin{aligned} \frac{\partial\vec{e}_\phi}{\partial\phi} &= \vec{e}_R \quad (= \vec{\nabla} R) \\ \frac{\partial\vec{e}_R}{\partial\phi} &= \vec{e}_\phi \quad (= R^2\vec{\nabla}\phi) \end{aligned}$$

and

$$\vec{e}_Z \times \vec{e}_\phi = R\cos\phi\,\vec{e}_x - R\sin\phi\,\vec{e}_y = R\vec{e}_R, \quad \vec{e}_\phi \times \vec{e}_R = R\vec{e}_Z, \quad \vec{R}_R \times \vec{e}_Z = R^{-1}\vec{e}_\phi$$

so that

$$\mathcal{J} = \vec{e}_R \cdot (\vec{e}_Z \times \vec{e}_\phi) = R$$

Notes

(R, Z, ϕ) coordinate system continued:

$$g_{i,j} = \begin{pmatrix} \vec{e}_R \cdot \vec{e}_R & \vec{e}_R \cdot \vec{e}_Z & \vec{e}_R \cdot \vec{e}_\phi \\ \vec{e}_Z \cdot \vec{e}_R & \vec{e}_Z \cdot \vec{e}_Z & \vec{e}_Z \cdot \vec{e}_\phi \\ \vec{e}_\phi \cdot \vec{e}_R & \vec{e}_\phi \cdot \vec{e}_Z & \vec{e}_\phi \cdot \vec{e}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R^2 \end{pmatrix}$$

Covariant identities:

$$\begin{aligned} \vec{e}^R &= \vec{\nabla} R = \mathcal{J}^{-1} \vec{e}_Z \times \vec{e}_\phi = \vec{e}_R \\ \vec{e}^Z &= \vec{\nabla} Z = \mathcal{J}^{-1} \vec{e}_\phi \times \vec{e}_R = \vec{e}_Z \\ \vec{e}^\phi &= \vec{\nabla} \phi = \mathcal{J}^{-1} \vec{e}_R \times \vec{e}_Z = R^{-2} \vec{e}_\phi, \quad |\vec{e}^\phi| = R^{-1} \end{aligned}$$

and

$$g^{i,j} = \begin{pmatrix} \vec{e}^R \cdot \vec{e}^R & \vec{e}^R \cdot \vec{e}^Z & \vec{e}^R \cdot \vec{e}^\phi \\ \vec{e}^Z \cdot \vec{e}^R & \vec{e}^Z \cdot \vec{e}^Z & \vec{e}^Z \cdot \vec{e}^\phi \\ \vec{e}^\phi \cdot \vec{e}^R & \vec{e}^\phi \cdot \vec{e}^Z & \vec{e}^\phi \cdot \vec{e}^\phi \end{pmatrix} = \begin{pmatrix} \vec{\nabla} R \cdot \vec{\nabla} R & \vec{\nabla} R \cdot \vec{\nabla} Z & \vec{\nabla} R \cdot \vec{\nabla} \phi \\ \vec{\nabla} Z \cdot \vec{\nabla} R & \vec{\nabla} Z \cdot \vec{\nabla} Z & \vec{\nabla} Z \cdot \vec{\nabla} \phi \\ \vec{\nabla} \phi \cdot \vec{\nabla} R & \vec{\nabla} \phi \cdot \vec{\nabla} Z & \vec{\nabla} \phi \cdot \vec{\nabla} \phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/R^2 \end{pmatrix}$$

Some vector calculus identities (more in Goedbloed book) in terms of

$$\vec{B} = B_R \vec{e}^R + B_Z \vec{e}^Z + B_\phi \vec{e}^\phi = B_R \vec{e}^R + B_Z \vec{e}^Z + \hat{B}_\phi \hat{\boldsymbol{e}}^\phi, \quad \hat{\boldsymbol{e}}_\phi = \frac{\hat{\boldsymbol{e}}^\phi}{|\hat{\boldsymbol{e}}^\phi|} = R \vec{e}^\phi = R^{-1} \vec{e}_\phi$$

$$\begin{aligned} \vec{\nabla} f &= \vec{e}^R \frac{\partial f}{\partial R} + \vec{e}^Z \frac{\partial f}{\partial Z} + \vec{e}^\phi \frac{\partial f}{\partial \phi} \\ &= \vec{e}_R \frac{\partial f}{\partial R} + \vec{e}_Z \frac{\partial f}{\partial Z} + \frac{\vec{e}_\phi}{R^2} \frac{\partial f}{\partial \phi} \\ &= \vec{e}_R \frac{\partial f}{\partial R} + \vec{e}_Z \frac{\partial f}{\partial Z} + \frac{\hat{\boldsymbol{e}}_\phi}{R} \frac{\partial f}{\partial \phi} \end{aligned}$$

Notes

(R, Z, ϕ) coordinate system continued:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{B} &= \frac{1}{R}\frac{\partial}{\partial R}(RB^R) + \frac{\partial}{\partial Z}B^Z + \frac{1}{R}\frac{\partial}{\partial \phi}(RB^\phi) \\
&= \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial}{\partial Z}B_Z + \frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{B_\phi}{R}\right) \\
&= \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial}{\partial Z}B_Z + \frac{1}{R}\frac{\partial}{\partial \phi}\hat{B}_\phi
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{B} &= -\left(\frac{1}{R}\frac{\partial B^Z}{\partial \phi} - R\frac{\partial B^\phi}{\partial Z}\right)\vec{e}_R - \frac{1}{R}\left(\frac{\partial (R^2 B^\phi)}{\partial R} - \frac{\partial B^R}{\partial \phi}\right)\vec{e}_Z - \frac{1}{R}\left(\frac{\partial B^R}{\partial Z} - \frac{\partial B^Z}{\partial R}\right)\vec{e}_\phi \\
&= -\left(\frac{1}{R}\frac{\partial B_Z}{\partial \phi} - \frac{1}{R}\frac{\partial B_\phi}{\partial Z}\right)\vec{e}^R - \frac{1}{R}\left(\frac{\partial B_\phi}{\partial R} - \frac{\partial B_R}{\partial \phi}\right)\vec{e}^Z - R\left(\frac{\partial B_R}{\partial Z} - \frac{\partial B_Z}{\partial R}\right)\vec{e}^\phi \\
&= -\left(\frac{1}{R}\frac{\partial B_Z}{\partial \phi} - \frac{\partial \hat{B}_\phi}{\partial Z}\right)\hat{\boldsymbol{e}}_R - \frac{1}{R}\left(\frac{\partial (R\hat{B}_\phi)}{\partial R} - \frac{\partial B_R}{\partial \phi}\right)\hat{\boldsymbol{e}}_Z - \left(\frac{\partial B_R}{\partial Z} - \frac{\partial B_Z}{\partial R}\right)\hat{\boldsymbol{e}}_\phi
\end{aligned}$$

where $\hat{\boldsymbol{e}}$ notation indicates unit vectors. Note we have $B_Z = B^Z$, $B_R = B^R$, $\boldsymbol{e}_R = \hat{\boldsymbol{e}}_R = \boldsymbol{e}^R$, $\boldsymbol{e}_Z = \hat{\boldsymbol{e}}_Z = \boldsymbol{e}^Z$, $R^{-1}B_\phi = \hat{B}_\phi = RB^\phi$, $R^{-1}\boldsymbol{e}_\phi = \hat{\boldsymbol{e}}_\phi = R\boldsymbol{e}^\phi$

In the main lecture slides, the field is written as $\boldsymbol{B} = B_R\boldsymbol{e}_R + B_Z\boldsymbol{e}_Z + B_\phi\boldsymbol{e}_\phi$, where $\boldsymbol{e}_\phi$ is a unit vector, so the main lecture slides use the results in these notes pages, but transform the notes with $\hat{B}_\phi \to B_\phi$ and $\hat{\boldsymbol{e}}_\phi \to \boldsymbol{e}_\phi$,

Notes

Axisymmetric non-orthogonal coordinate system (r, Θ, ϕ)

It is possible to conveniently construct the metric for this system specifically for the case of axisymmetry via $R = R(r, \Theta)$ and $Z = Z(r, \Theta)$ and lifting some of the results from the notes on the (R, Z, ϕ) coordinates. Specifically, The cartesian system $\vec{c} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ is transformed to flux coordinates via

$$\begin{aligned} x &= R(r, \Theta)\cos\phi \\ y &= -R(r, \Theta)\sin\phi \\ z &= Z(r, \Theta) \end{aligned}$$

It is then possible to construct contravariant and covariant identities and associated vector (and tensor) calculus in terms of as yet undefined $R(r, \Theta)$. These notes are completed in the exercise series and associated worked solutions provided as part of this course.