Plasma Instabilities

Exercises Series 5

Linear and non-linear Tearing Modes

Autumn Semester 2025

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Questions on non-linear island width

1. In the course slides the equation for the helical field trajectory near a rational surface is defined as:

$$x^2 = \frac{2}{r_s s B_0^\theta}\left[\Psi - \frac{r_s \hat{B}_1^r(t)}{m}\cos(m\chi)\right],$$

all evaluated at r_s. It is stated that the field line of the separatrix is mapped out for a total flux Ψ matching $r_s \hat{B}_1^r(r_s, t)/m$. Defining the island width w as width of the separatrix at the angle that makes x maximum, show that,

$$w(t) = 4r_s\left(\frac{\hat{B}_1^r(t)}{msB_0^\theta}\right)^{1/2}.$$

2. starting with

$$\int_{r_s-w/2}^{r_s+w/2} dr \frac{\partial \Psi_1}{\partial t} = \eta \frac{\partial \Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2}$$

Use the final result from the previous question and the constant-psi approximation to obtain the Rutherford equation

$$\frac{dw}{dt} = \frac{\eta(r_s)}{2}\Delta'(w) \quad \text{with} \quad \Delta'(w) = \frac{1}{\Psi_1}\frac{d\Psi_1}{dr}\bigg|_{r_s-w/2}^{r_s+w/2}.$$

3. Extend the previous question to include non-Ohmic current perturbations, such as bootstrap current effects or auxiliary current drive effects. Start with

$$\frac{\partial \Psi_1}{\partial t} = \eta\left[\frac{\partial^2 \Psi_1}{\partial r^2} + j_{non}\right] \qquad (1)$$

where $j_{non}(r) = j_{BS}(r) + j_{cd}(r)$ is the sum of e.g. bootstrap and auxiliary current drive, both are toroidal currents. Assume directly the constant-psi approximation, by allowing radial dependence in Eq. (1) only in Ψ_1''. Then integrate Eq. (1) with respect to r over $[r_s - w/2, r_s + w/2]$ to eventually obtain:

$$\frac{dw(t)}{dt} = \frac{\eta(r_s)}{2}\left[\Delta'(w) + \Delta'_{BS}(w) + \Delta'_{cd}(w)\right], \quad \text{with} \quad \Delta'_X(w) = \frac{j_X}{w}\frac{16r}{sB_0^\theta}\bigg|_{r_s}.$$

Why is the result in the literature usually written as:

$$\frac{dw(t)}{dt} = \frac{\eta}{2}\left[\Delta'(w) + \Delta'_{BS}(w) + \Delta'_{cd}(w)\right], \quad \text{with} \quad \Delta'_X(w) = j_X\frac{16r}{sB_0^\theta}\left(\frac{w}{w_c^2 + w^2}\right)\bigg|_{r_s},$$

with w_c a constant.

Questions on linear tearing mode solution in the layer, and the layer width

4. What is the value of $\delta\psi$ on the rational surface under the ideal MHD limit if ξ^r is not singular on the rational. Consider Eq. (3). What if instead $\xi^r(x) \sim 1/x$ across $x = 0$, where $x = (r - r_s)/r_s$. Is your answer consistent with the idea that only resistive modes can change the topology of the magnetic field (relative to the equilibrium magnetic field)?

5. Use e.g. matlab or Mathematica to numerically integrate $y(z)$, where $y \propto \xi_0^r/\delta\psi(r_s)$ is defined in the lecture notes. Numerically integrate:
$$y = \frac{z}{2}\int_0^1 d\mu \frac{\exp(-z^2\mu/2)}{(1-\mu^2)^{1/4}}.$$
Show that it is odd, has a dipole structure, its asymptote is $y = 1/z$ for $|z| \gtrapprox 1$. Here, $y = 1/z$ is the ideal current sheet solution. Hence, if we define the layer width as the width over which $y(z)$ is different from the inertialess ideal solution, show that the layer width is $\approx 2\delta$, where we note that,
$$z = \frac{r_s x}{\delta}.$$

6. We expect that the layer width conforms to $(\xi^r)'' \sim \xi^r/\delta^2$. The displacement varies rapidly across the layer, even if $\delta\psi$ does not. Show that your solution to $y(z)$ is consistent with this.

7. In the lecture notes it is shown that assuming that $(\xi^r)'' \sim \xi^r/\delta^2$ (which we verified in the previous question) then,
$$\delta\,\Delta' \ll 1 \qquad (2)$$
must hold in order to adopt the constant-psi approximation in the calculation of the dispersion relation for tearing modes. From the two equations given in the lecture notes,
$$\frac{\delta}{r_s} = \frac{1}{d} = \left[\frac{\gamma}{\omega_A n^2 s^2 S}\right]^{1/4}$$
$$\frac{\gamma}{\omega_A} = \left[\frac{\Gamma(1/4) r_s \Delta'}{2\pi\Gamma(3/4)}\right]^{4/5} S^{-3/5}(ns)^{2/5}$$
obtain
$$\delta\,\Delta' = 2.12\frac{S^{1/2}}{ns}\left(\frac{\gamma}{\omega_A}\right)^{3/2}$$
and hence for the inequality of Eq. (2) we require,
$$\frac{\gamma}{\omega_A} \ll \frac{(ns)^{2/3}}{S^{1/3}}.$$
Assume $S = 10^8$, $ns = 1$, how small should γ/ω_A be for Eq. (2) to apply, and is this reasonable for resistive modes if ideal modes tend to be of order $\gamma/\omega_A \sim 10^{-2} - 10^{-3}$.

8. Eliminate the growth rate using the equations defined in the previous question to obtain,
$$\frac{\delta}{r_s} = S^{-2/5}(ns)^{-2/5}\left(\frac{r_s\Delta'}{2.12}\right)^{1/5}$$
Recall that Δ' is obtained from the external solutions. It is found that for reasonable tokamak like q-profiles $|r_s\Delta'| \sim 1$. Hence show that,
$$\frac{\delta}{r_s} \sim S^{-2/5}(ns)^{-2/5}$$
which is indeed small. What is the order of magnitude of δ/r_s for the example values of S and ns given in the previous question assuming $r_s\Delta' = +1$. Does this confirm $\delta\,\Delta' \ll 1$ required for constant-psi approximation? What is the expected order of magnitude for γ/ω_A?

9. We calculate Δ' in the layer via the asymptotic form:

$$\Delta' = \frac{1}{\delta\psi(x=0)}\left[\lim_{X\to\infty}\delta\psi'(X) - \lim_{X\to-\infty}\delta\psi'(X)\right].$$

In the slides, it is shown that Δ' in the layer is proportional to

$$\lim_{X\to\infty}\Delta'_X, \quad \text{with} \quad \Delta'_X = \int_{-X}^{X} dz\,[1 - zy(z)].$$

Using your numerical solution for $y(z)$ show that Δ'_X saturates to nearly constant amplitude for $X \gtrapprox 4$, and that it tends towards the asymptotic value $2\pi\Gamma(3/4)/\Gamma(1/4)$ for $X \to \infty$. Hence show that this saturation for $|x| \gtrapprox \delta$ validates matching with the asymptotic behaviour of the layer (with the ideal region).

Question on the *outer* equations

10. Show that the equation for $\delta\psi$ for tearing instability calculation of the magnetic field in the outer region (region where we can neglect inertia and resistivity):

$$r\frac{d}{dr}\left(r\frac{d\delta\psi}{dr}\right) + \left(\frac{R_0}{B_0}\right)\frac{rqm\delta\psi}{nq - m}\frac{dJ_\phi}{dr} - m^2\delta\psi = 0.$$

and also the equation used for external kink calculation of the displacement,

$$r^2\frac{d^2\xi_0^r}{dr^2} + r\frac{d\xi_0^r}{dr}\left[3 - \frac{2s(r)}{1 - \frac{nq(r)}{m}}\right] - (m^2 - 1)\xi_0^r = 0$$

are equivalent to the Euler-Lagrange equation obtained in week 3 from the variation of δW_2:

$$\frac{d}{dr}\left[\left(\frac{n}{m} - \frac{1}{q}\right)^2 r^3\frac{d\xi_0^r}{dr}\right] = (m^2 - 1)\left(\frac{n}{m} - \frac{1}{q}\right)^2 r\xi_0^r.$$

To answer this question you may want to verify that,

$$J_\parallel \approx J_\phi = \frac{1}{r}\frac{d}{dr}(rB_p) \approx \frac{B_0}{R_0}\frac{1}{r}\frac{d}{dr}\left(\frac{r^2}{q}\right)$$

and you will need to use the ideal (outer region) limit of:

$$\xi_0^r = \frac{R_0}{B_0 r}\left(\frac{n}{m} - \frac{1}{q}\right)^{-1}\left[\delta\psi - \frac{r\eta}{\gamma}\nabla^2\left(\frac{\delta\psi}{r}\right)\right] \tag{3}$$

(this latter equation in the ideal limit was obtained in week 3, calculation of the radial magnetic field in terms of ξ^r).