Plasma Instabilities

Exercises Series 3

Theory of linear ideal MHD

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From question number 2 onwards this exercise is based on straight field line coordinates (r, θ, ϕ) with $\mathcal{J}_\theta = rR^2/R_0$.

Relevant equilibrium properties

1. Consider the curvature vector

$$\boldsymbol{\kappa} = (\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b}, \quad \boldsymbol{b} = \frac{\boldsymbol{B}}{B}$$

   which is proportional to the magnetic operator $\boldsymbol{B}\cdot\boldsymbol{\nabla}$ operating on $\boldsymbol{b}$. Using equilibrium force balance $\boldsymbol{J}\times\boldsymbol{B} = \boldsymbol{\nabla}P$, Amperes law, and well know vector calculus identities, show that

$$\frac{1}{B^2}\boldsymbol{\nabla}_\perp\left(\frac{B^2}{2}+P\right) = (\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b},$$

   where

$$\boldsymbol{\nabla}_\perp = \boldsymbol{\nabla} - \boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla}),$$

   and hence that

$$\kappa = \frac{\boldsymbol{\nabla}_\perp}{B^2}\left(\frac{B^2}{2}+P\right). \tag{1}$$

2. It is shown in the lecture notes that part of δW comprises $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}$, and δW is zero at order ϵ^0 and order ϵ^2 by setting $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} = 0$. On assuming $P/B^2 \sim \epsilon^2$ and $RB = R_0B_0(1+O(\epsilon^2))$ (see exercise series 2), show that when applying straight field line coordinates, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} = \frac{\mathcal{J}_\theta}{r}\boldsymbol{\nabla}\cdot\left(\frac{r\boldsymbol{\xi}_\perp}{\mathcal{J}_\theta}\right)\left(1+O(\epsilon^2)\right) \tag{2}$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(r\xi^r) + \frac{\partial}{\partial\theta}\left(\xi_\perp^\theta\right) + \frac{r}{R}\frac{\partial}{\partial\phi}\left(\xi_\perp^\phi\right)\right](1+O(\epsilon^2)), \tag{3}$$

   where

$$\xi^r = \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}r, \quad \xi_\perp^\theta = r\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\theta, \quad \xi_\perp^\phi = R\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\phi.$$

   Hint: use result from previous question (definition of curvature) and start with the general definition of divergence (in terms of Jacobian) as defined in the lectures.

3. For the representation of the field $\boldsymbol{B} = F(r)\boldsymbol{\nabla}\phi + \psi'\boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}r$ show that when adopting straight field line coordinates (r, θ, ϕ) the **magnetic operator** $\boldsymbol{B}\cdot\boldsymbol{\nabla}$ can be written at:

$$\boldsymbol{B}\cdot\boldsymbol{\nabla} = \frac{F}{R^2}\left[\frac{\partial}{\partial\phi} + \frac{1}{q}\frac{\partial}{\partial\theta}\right]. \tag{4}$$

   And therefore that:

$$\frac{\partial}{\partial l} \equiv \boldsymbol{b}\cdot\boldsymbol{\nabla} = \frac{F}{BR^2}\left[\frac{\partial}{\partial\phi} + \frac{1}{q}\frac{\partial}{\partial\theta}\right]$$

where $\boldsymbol{b} = \boldsymbol{B}/B$. Note, when operating on fluctuations, the parallel wavenumber $k_{\|}$ can be identified via $\partial/\partial l = \boldsymbol{b}\cdot\boldsymbol{\nabla} = ik_{\|}$

Properties of vector displacements and perturbed fields

The linear ideal MHD equations can be written in terms of the displacement variable $\boldsymbol{\xi} = \boldsymbol{\xi}_{\perp} + \xi_{\|}\boldsymbol{b}$. The perturbed magnetic field is,

$$\boldsymbol{\delta B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}) = \boldsymbol{\nabla}\times(\boldsymbol{\xi}_{\perp}\times\boldsymbol{B}) \tag{5}$$

We write the covariant and contravariant perpendicular displacements respectively in the form:

$$\boldsymbol{\xi}_{\perp} = \xi_r\boldsymbol{\nabla} r + r\xi_{\perp\theta}\boldsymbol{\nabla}\theta + R\xi_{\perp\phi}\boldsymbol{\nabla}\phi$$

and

$$\xi^r = \boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\nabla} r, \quad \xi^{\theta}_{\perp} = r\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\nabla}\theta, \quad \xi^{\phi}_{\perp} = R\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\nabla}\phi.$$

(with these definitions $\xi^{\phi}_{\perp} = \xi_{\perp\phi}$ and $\xi^{\theta}_{\perp} = \xi_{\perp\theta}(1+O(\epsilon))$, where the correction in the latter is due to the non-orthogonality of the coordinate system. We may also write,

$$\xi_{\|}\boldsymbol{b} = r\xi_{\|\theta}\boldsymbol{\nabla}\theta + R\xi_{\|\phi}\boldsymbol{\nabla}\phi$$

and

$$\xi^{\theta}_{\|} = r\xi_{\|}\boldsymbol{b}\cdot\boldsymbol{\nabla}\theta, \quad \xi^{\phi}_{\|} = R\xi_{\|}\boldsymbol{b}\cdot\boldsymbol{\nabla}\phi.$$

4. Show that

$$\xi_{\perp\phi} = -\frac{r}{qR}\xi_{\perp\theta}$$

(notice that $\xi_{\perp}$ has only two components (and one of them is ξ^r), so it is clear that $\xi^{\phi}_{\perp}$ and $\xi^{\theta}_{\perp}$ would not be independent). Less important for these exercises, but you might also show that

$$\xi_{\|\phi} = \xi^{\phi}_{\|} = \frac{F}{RB}\xi_{\|}, \quad \text{and} \quad \xi_{\|\theta} = \frac{r}{qR}\xi_{\|\phi}.$$

5. From your answer to the last question show that

$$\xi^{\phi}_{\perp} = -\frac{\epsilon}{q}\xi^{\theta}_{\perp}(1+O(\epsilon)), \quad \epsilon = \frac{r}{R_0}$$

And from this result, show using Eq. (3) in question 2 that,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_{\perp} + 2\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\kappa} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\xi^r) + \frac{\partial}{\partial\theta}\left(\xi^{\theta}_{\perp}\right)\right](1+O(\epsilon^2)). \tag{6}$$

where it is assumed $\partial/\partial\phi \sim -q^{-1}\partial/\partial\theta$ which of course forces $\partial/\partial l \to 0$ (as expected for resonant instabilities with dominant poloidal mode number m and toroidal mode number n, and resonant surface r where $q(r) = m/n$)

This result is important because δW involves $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_{\perp} + 2\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\kappa}$, and in the order expansion developed in δW_0 and δW_2, it can be shown that

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_{\perp} + 2\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\kappa} = 0. \tag{7}$$

In the expressions for δW_0 and δW_2 it is legitimate to drop the $O(\epsilon^2)$ correction in Eq. (6) because those corrections clearly first appear in δW_4.

6. From the definition (see lecture notes), we have that

$$\boldsymbol{\xi}_{\perp 0}(r,\theta,\phi) = \hat{\boldsymbol{\xi}}_{\perp 0}(r)\exp(-im\theta + in\phi)$$

Show using the result of Eq. (6) and Eq. (7) that

$$\xi^{\theta}_{\perp 0} = -\frac{i}{m}\frac{\partial}{\partial r}\left(r\xi^r_0\right).$$

7. From the definition of Eq. (5), show using vector calculus identities that,

$$\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}X = \boldsymbol{\nabla}\cdot\left[(\boldsymbol{\xi}_\perp \times \boldsymbol{B})\times\boldsymbol{\nabla}X\right]$$

and furthermore that

$$\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}X = (\boldsymbol{B}\cdot\boldsymbol{\nabla})(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}X) - \boldsymbol{\nabla}\cdot\left[\boldsymbol{\xi}_\perp\,(\boldsymbol{B}\cdot\boldsymbol{\nabla}X)\right] \tag{8}$$

8. Use Eq. (8) to show that

$$\delta B^r \equiv \boldsymbol{\delta B}\cdot\boldsymbol{\nabla}r = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\xi^r.$$

Notice that δB^r is defined as the magnetic operation on ξ^r.

Show, using the results of the previous questions in this series that to lowest order (in ϵ)

$$\delta B^r_0 = \frac{iB_0}{R_0 q}\left[nq - m\right]\xi^r_0$$

using also that $F = R_0 B_0(1 + O(\epsilon^2))$. Comment on what happens to δB^r_0 on a rational surface $q(r) = m/n$.

9. Use again Eq. (8) to show that

$$\delta B^\theta \equiv r\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}\theta = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\xi^\theta_\perp - \frac{rF}{qR_0}\boldsymbol{\nabla}\cdot\left(\frac{r\boldsymbol{\xi}_\perp}{\mathcal{J}_\theta}\right) - \frac{r}{R^2}\xi^r\frac{d}{dr}\left(\frac{F}{q}\right),$$

where

$$\boldsymbol{\nabla}\cdot\left(\frac{r\boldsymbol{\xi}_\perp}{\mathcal{J}_\theta}\right) = \frac{1}{\mathcal{J}_\theta}\left[\frac{\partial}{\partial r}\left(r\xi^r\right) + \frac{\partial}{\partial\theta}\left(\xi^\theta_\perp\right) + \frac{r}{R}\frac{\partial}{\partial\phi}\left(\xi^\phi_\perp\right)\right].$$

Adopt the previous results of this question sheet to find that to lowest order,

$$\delta B^\theta_0 = \frac{B_0}{R_0 q}\left[s(r)\xi^r_0 + \frac{nq - m}{m}\frac{\partial}{\partial r}\left(r\xi^r_0\right)\right].$$

where

$$s = \frac{r}{q}\frac{dq}{dr}$$

is the magnetic shear.

10. Following a similar approach to the last questions it can be shown that,

$$\delta B^\phi \equiv R\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}\phi = R(\boldsymbol{B}\cdot\boldsymbol{\nabla})(\xi^\phi_\perp/R) - \frac{RF}{R_0}\boldsymbol{\nabla}\cdot\left(\frac{r\boldsymbol{\xi}_\perp}{\mathcal{J}_\theta}\right) - \frac{1}{R}\xi^r\frac{dF}{dr}$$

which is clearly very small on a rational surface. Use this result and the ordering (in ϵ) of non-orthogonal coordinate corrections, to argue that,

$$\left|\delta B_{\perp 0}\right|^2 = \left|\delta B^r_0\right|^2 + \left|\delta B^\theta_0\right|^2$$

and use the previous two questions to consider $|\delta B_{\perp 0}|^2$ on a rational surface. Does it vanish on a rational surface if the magnetic shear is non-zero? By considering magnetic field line bending stabilisation (see lecture notes), why do instabilities tend to align to rational surfaces? What is the parallel wavenumber associated with leading order fluctuations on the rational surface? Show that stability can be improved if configuration has non-zero magnetic shear.

11. We now calculate $\delta B_{\|} = \boldsymbol{b} \cdot \boldsymbol{\delta B}$ in terms of the perpendicular displacement. Show using Amperes law and force balance $\boldsymbol{J} \times \boldsymbol{B} = \boldsymbol{\nabla} P$ that,

$$\delta B_{\|} = -B \left[\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_{\perp} + 2\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa} \right] + \frac{\xi^r}{B} \frac{dP}{dr}.$$

Can this be reduced further for realistic MHD instabilities (ones that minimise δW), and does it follow that parallel magnetic fluctuations are directly associated with finite beta effects?