Solutions Problem Sheet 11: Fidelity

Text Book

Many solutions in this problem sheet can be found in the standard text books. We will refer to some parts of "Quantum Computation and Quantum Information" 10th edition by Michael A. Nielsen and Isaac L. Chuang, which can be found online for free here, or by googling "Quantum Computation and Quantum Information".

Class problems

1. Compute the fidelity between the following pairs of states:

   a) $\rho = |\Phi_+\rangle$ and $\sigma = |00\rangle$

   b) $\rho = 3/4|\Phi_+\rangle\langle\Phi_+| + 1/4|\Psi_-\rangle\langle\Psi_-|$ and $\sigma = |00\rangle$.

   c) $\rho = 3/4|\Phi_+\rangle\langle\Phi_+| + 1/4|\Psi_-\rangle\langle\Psi_-|$ and $\sigma = 3/7|00\rangle\langle 00| + 4/7|++\rangle\langle ++|$

   Answers 1: For a), the fidelity (as defined in the Nielsen and Chuang's book) is $|\langle\Phi_+|00\rangle| = 1/\sqrt{2}$.
   For b) and c), consider the fidelity for general quantum states ρ and σ.

$$F(\rho,\sigma) = \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right] . \tag{1}$$

   To compute the fidelity, the strategy is to pick ρ in a smart way such that $\sqrt{\rho}$ is easy to be computed – this is usually either a pure quantum state or a state that is already in a diagonal form. For b), we pick $\rho = |00\rangle\langle 00|$, leading to $\sqrt{\rho} = |00\rangle\langle 00|$ and

$$F(\rho,\sigma) = \sqrt{\langle 00|\left(3/4|\Phi_+\rangle\langle\Phi_+| + 1/4|\Psi_-\rangle\langle\Psi_-|\right)|00\rangle} \tag{2}$$

$$= \sqrt{\frac{3}{8}} . \tag{3}$$

   For c), we pick $\rho = 3/4|\Phi_+\rangle\langle\Phi_+| + 1/4|\Psi_-\rangle\langle\Psi_-|$ which is diagonal in the Bell basis and corresponds to $\sqrt{\rho} = \sqrt{3}/2|\Phi_+\rangle\langle\Phi_+| + 1/2|\Psi_-\rangle\langle\Psi_-|$. The quantum fidelity can be computed as

$$F(\rho,\sigma) = \mathrm{Tr}\left[\sqrt{\left(\sqrt{3}/2|\Phi_+\rangle\langle\Phi_+| + 1/2|\Psi_-\rangle\langle\Psi_-|\right)\left(3/7|00\rangle\langle 00| + 4/7|++\rangle\langle ++|\right)\left(\sqrt{3}/2|\Phi_+\rangle\langle\Phi_+| + 1/2|\Psi_-\rangle\langle\Psi_-|\right)}\right] \tag{4}$$

$$= \mathrm{Tr}[\sqrt{3/8|\Phi_+\rangle\langle\Phi_+|}] \tag{5}$$

$$= \sqrt{3/8} \tag{6}$$

   where we use the fact that $\langle 00|\Phi_+\rangle = \langle ++|\Phi_+\rangle = 1/\sqrt{2}$ and $\langle 00|\Psi_-\rangle = \langle ++|\Psi_-\rangle = 0$.

2. We will keep using this polar decomposition throughout this class problem. So, here is some kick-start for you to prove it.

   a) Show that any complex square matrix A can be expressed in the polar decomposition $A = |A|U$ where $|A| = \sqrt{AA^\dagger}$ and U is some unitary (Hint - singular value decomposition).

   b) Show that $\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = \sqrt{\rho}\sqrt{\sigma}\tilde{V}$ for some unitary $\tilde{V}$.

Answers 2: For a), Consider the singular value decomposition of A which has the form $A = UDW^\dagger$ with D as some positive diagonal matrix and U, W as some unitaries. Notice that $\sqrt{AA^\dagger} = \sqrt{UD^2U^\dagger} = UDU^\dagger$. Therefore, we can write $A = UDW^\dagger = UD(U^\dagger U)W^\dagger = |A|V$ with $V = UW^\dagger$.

For b), consider $A = \sqrt{\rho}\sqrt{\sigma}$ then $|A| = \sqrt{AA^\dagger} = \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$. Using the polar decomposition we have $A = |A|V \to AV^\dagger = |A|$. Hence, $\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\tilde{V} = \sqrt{\rho}\sqrt{\sigma}$ with $\tilde{V} = V^\dagger$.

3. This question works you through the proof of Uhlmann's theorem.

a) Use Cauchy–Schwarz inequality to show that $\mathrm{Tr}[AU] \leqslant \mathrm{Tr}|A|$ where we write A in terms of its polar decomposition as $A = |A|V$. In addition, verify that equality is attained with $U = V^\dagger$.

b) Why can the purification of a state ρ be written as $|\psi\rangle = U_R \otimes \sqrt{\rho}U_S|\mathrm{Vec}(\mathbb{I})\rangle$? Where U_S and U_R are unitaries. (Hint: Schmidt decomposition).

c) Show that $\langle \mathrm{Vec}(\mathbb{I})|A^* \otimes B|\mathrm{Vec}(\mathbb{I})\rangle = \mathrm{Tr}[A^\dagger B]$

d) Hence show that $\max_{|\psi\rangle,|\phi\rangle} |\langle\psi|\phi\rangle| = \mathrm{Tr}[\sqrt{\rho^{1/2}\sigma\rho^{1/2}}] = \mathrm{F}(\rho,\sigma)$ where the maximisation is taken over all purifications $|\psi\rangle$ and $|\phi\rangle$ of ρ and σ.

Answers 3: The whole proof can be found in p. 410-411 in the Nielsen and Chuang's book. The steps to do a) and d) are explained in details, however the solutions for b) and c) are not presented in the book. We will be doing b) and c) here.

For b), consider the eigendecomposition of a state as $\rho = \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|$. Any purified state of ρ must respect the following form of the Schmidt decomposition

$$|\Psi\rangle = \sum_k \sqrt{\lambda_k}|\Phi_k\rangle \otimes |\lambda_k\rangle \ , \tag{7}$$

where $\{|\Phi_k\rangle\}$ are orthogonal states in the extended system. We now verify that $|\psi\rangle = U_R \otimes \sqrt{\rho}U_S|\mathrm{Vec}(\mathbb{I})\rangle$ indeed respects the form in Eq. (7). By plugging $\sqrt{\rho} = \sum_k \sqrt{\lambda_k}|\lambda_k\rangle\langle\lambda_k|$ and $|\mathrm{Vec}(\mathbb{I})\rangle = \sum_i |i\rangle \otimes |i\rangle$, we have

$$U_R \otimes \sqrt{\rho}U_S|\mathrm{Vec}(\mathbb{I})\rangle = U_R \otimes \left(\sum_k \sqrt{\lambda_k}|\lambda_k\rangle\langle\lambda_k|\right) U_S \left(\sum_i |i\rangle \otimes |i\rangle\right) \tag{8}$$

$$= \sum_k \sqrt{\lambda_k}\left(\sum_i \langle\lambda_k|U_S|i\rangle U_R|i\rangle\right) \otimes |\lambda_k\rangle \tag{9}$$

$$= \sum_k \sqrt{\lambda_k}|\tilde{\Phi}_k\rangle \otimes |\lambda_k\rangle \ , \tag{10}$$

Notice that $\{|\tilde{\Phi}_k\rangle\}$ are indeed orthogonal states. That is, by using the identity $\mathbb{1} = \sum_i |i\rangle\langle i|$, we have $\langle\tilde{\Phi}_k|\tilde{\Phi}_{k'}\rangle = \langle\lambda_k|\lambda_{k'}\rangle = \delta_{k,k'}$. Therefore, it respects the generic form of the purified state.

For c), we explicitly expand the expression as

$$\langle \mathrm{Vec}(\mathbb{I})|A^* \otimes B|\mathrm{Vec}(\mathbb{I})\rangle = \sum_{i,j} (\langle j| \otimes \langle j|)(A^* \otimes B)(|i\rangle \otimes |i\rangle) \tag{11}$$

$$= \sum_{i,j} \langle j|A^*|i\rangle\langle j|B|i\rangle \tag{12}$$

$$= \sum_{i,j} \langle i|A^\dagger|j\rangle\langle j|B|i\rangle \tag{13}$$

$$= \mathrm{Tr}[A^\dagger B] \ , \tag{14}$$

where in the third equality we use the fact that $\langle j|A^*|i\rangle = \langle i|A^\dagger|j\rangle$ and in the last equality we use $\mathbb{1} = \sum_j |j\rangle\langle j|$.

4. Use Uhlmann's theorem to show that the data processing inequality holds for quantum fidelity, i.e. $\mathrm{F}(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geqslant \mathrm{F}(\rho, \sigma)$ for any trace preserving quantum operation $\mathcal{E}$.

   Answers 4: The proof can be found in p. 414 in the Nielsen and Chuang's book.

5. This question works you through the derivation of the following operational expression for the fidelity

$$F(\rho, \sigma) = \mathrm{Tr}[\sqrt{\rho^{1/2}\sigma\rho^{1/2}}] = \min_{\{M_i\}} \sum_i \sqrt{\mathrm{Tr}[\rho M_i]\mathrm{Tr}[\sigma M_i]}\,. \tag{15}$$

   a) Show that $\mathrm{Tr}[\sqrt{\rho^{1/2}\sigma\rho^{1/2}}] = \sum_i \mathrm{Tr}[\sqrt{\rho}\sqrt{M_i}\sqrt{M_i}\sqrt{\sigma}V]$ for a set of POVMs $\{M_i\}$ where V is some unitary (Hint - polar decomposition).

   b) Hence use Cauchy-Schwarz to show Eq. (15).

   Answers 5: The proof can be found in p. 412 in the Nielsen and Chuang's book.