

# Problem Sheet 9: Distinguishing quantum states and Matrix norms

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

## Class problems

1. Suppose you are given a state  $\rho_0$  with probability  $p$  and a state  $\rho_1$  with probability  $1 - p$  (single qubit states).
  - a) What is the optimum binary POVM to answer the question ‘Is the system in the state  $\rho_0$  or  $\rho_1$ ’?
  - b) What is the probability of successfully distinguishing  $\rho_0$  and  $\rho_1$ ?
2. Show that the Schatten  $p$ -norms are unitarily invariant. That is,  $\|A\|_p = \|UAV\|_p$  for any unitaries  $U$  and  $V$ .
3. Write  $\|Q\|_\infty$  in terms of the eigenvalues of  $Q$  (you may assume  $Q$  is square).
4. Use the unitary invariance of the Schatten  $p$ -norms and the triangle inequality to show that

$$\|U^N - V^N\|_p \leq N\|U - V\|_p.$$

5. Show that  $\|Q\|_1 \geq \|Q\|_2 \geq \|Q\|_\infty$ .
6. Now for any  $p, q \geq 1$ , show that  $\|Q\|_p \leq \|Q\|_q \iff p \geq q$ . Notice that  $p > q$  does not imply that  $\|Q\|_p > \|Q\|_q$ . (Hint: you can use the fact that for real positive  $x_i$  and  $r \geq 0$ , we have  $\sum_i x_i^{1+r} \leq (\sum_i x_i)^{1+r}$ . If you use it, try at least to show it.).
7. Write (/find in built functions) in Python and Mathematica code that can compute the Schatten 1, 2, and infinity norms of a (Haar) random unitary matrix. What is the average distance (for each norm) between two randomly chosen unitaries.