Problem Sheet 3

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Purifications

1. Compute purifications for the following states:

   a) $\rho_1 = 1/2(|0\rangle\langle 0| + |1\rangle\langle 1|)$
   b) $\rho_2 = 1/2(|0\rangle\langle 0| + |+\rangle\langle +|)$
   c) $\rho_3 = 1/2(|\psi_+\rangle\langle \psi_+| + |\phi_-\rangle\langle \phi_-|)$

2. Consider the single qubit state $\rho = \frac{1}{2}(\mathbb{I} + 0.1X + 0.1Y + 0.2Z)$

   a) Write ρ as a matrix in the computational basis.

   b) Compute the eigen-decomposition of ρ.

   c) Is ρ mixed or pure? How do you know?

   d) Compute a pure state decomposition of ρ involving three states.

   e) Hence state i. a purification of ρ using a single qubit environment and ii. a purification using a qutrit environment.

Measurements

3. Explain what is the difference between POVM measurements, projective measurements and a measurement of an observable.

4. a) Write down a POVM measurement $\mathcal{M}$ that asks the question: "Is the system in the $|\Phi_+\rangle$ Bell state?"

   b) Consider the 2-qubit separable state $|\psi\rangle \otimes |\psi\rangle$. What is the probability to find the system in $|\Phi_+\rangle$?

5. Consider a d dimensional system S and $\mathcal{M} = \{\alpha_0 \mathbb{I}, \alpha_1 \mathbb{I}, \alpha_2 \mathbb{I}, \alpha_3 \mathbb{I}\}$. Is this a valid measurement? If so, what does the measurement do?

6. Suppose Bob hands you a quantum state and promises that it is either $|\phi_1\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$ or $\cos(\theta/2)|0\rangle - \sin(\theta/2)|1\rangle$.

a) Sketch these states on the Bloch sphere.

b) Design a measurement that perfectly distinguishes the states *some* of the time, is inconclusive at others, but never makes a mistake.

c) What is the probability in a *single* run of the experiment that you guess correctly?

d) Write down a projective measurement on a larger system that can be used to realise this POVM.

7. Propose an 'informationally complete' measurement for a single qubit state. That is, a POVM measurement $\mathcal{M}$ that allows you to perfectly reconstruct a single qubit quantum state. What about a 2-qubit state?

8. Suppose $\mathcal{M} = \{M_i\}_{i=1}^m$ and $\mathcal{N} = \{N_i\}_{i=1}^n$ are two different POVM measurements.

a) We can define an $m + n$ outcome POVM from $\mathcal{M}$ and $\mathcal{N}$ by flipping a biased coin and with probability p doing $\mathcal{M}$ and probability $(1 - p)$ doing $\mathcal{N}$. Write down the measurement operators for this measurement.

b) Suppose now $m = n$. We can alternatively define a m measurement composed of the operators $\{pM_i + (1 - p)N_i\}_{i=1}^m$. How does this measurement differ from the one in part (a)? How could you realise it?