Problem Sheet 12: Pure state bipartite entanglement

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Class problems

Resource Theories.

1. a) Argue that any resource theory defines a partial order $\rho \preceq \sigma$ on the set of all quantum states $\mathcal{D}$ based on whether or not we can send $\rho \to \sigma$ under some free operation $\mathcal{E} \in F$.

   b) How can the notion of a resource measure M be defined using this partial ordering?

   c) Show that any two free states σ_1 and σ_2 are equal under the partial ordering.

   d) Show that any resource measure M must ascribe the same value to any two states that are equal under this ordering. (Does the converse hold?)

2. Make up your own resource theory! (Optional)

3. Prove that it is possible for Bob to perform a local operation conditional on an outcome of Alice's measurement via LOCC. That is, show it is possible to perform $\rho_{AB} \to \sum_i (M_i \otimes U_i)\rho_{AB}(M_i^\dagger \otimes U_i^\dagger)$ via LOCC.

4. Prove that product states are the states that can be prepared freely via LOCC.

Majorization.

1. Draw the partial order defined by majorization on the following vectors $\boldsymbol{v}_1 = (0, 1/3, 2/3)$, $\boldsymbol{v}_2 = (1/3, 1/3, 1/3)$, $\boldsymbol{v}_3 = (1/4, 1/5, 1/3)$, $\boldsymbol{v}_4 = (1/5, 3/5, 1/5)$, $\boldsymbol{v}_5 = (2/5, 1/2, 1/10)$, and $\boldsymbol{v}_6 = (0, 1, 0)$.

2. Show that $(1/N, 1/N, ..., 1/N) \preceq \boldsymbol{p} \preceq (1, 0, ..., 0)$ for any probability distribution $\boldsymbol{p}$.

3. A useful equivalent definition of majorization is that $\boldsymbol{x} \preceq \boldsymbol{y}$ iff $\boldsymbol{x}$ is a convex combination of vectors obtained by permuting coordinates of $\boldsymbol{y}$.

   Use this fact to show that the diagonal elements of a density operator are majorized by its eigenvalues. (Super useful property!)

   (Hint use the fact that any doubly stochastic matrix can be written as a convex combination of permutation matrices).

Bipartite Entanglement.

1. Argue that $|\psi_-\rangle_{AB}$ can be transformed into any state $|\phi\rangle_{AB}$ via LOCC using Nielson's Majorization Theorem. Describe a protocol to do this in practise.

2. Show that transforming between the states $|\phi\rangle = \sqrt{\frac{15}{100}}|00\rangle + \sqrt{\frac{3}{10}}|11\rangle + \sqrt{\frac{4}{10}}|22\rangle + \sqrt{\frac{15}{100}}|33\rangle$ and $|\psi\rangle = \sqrt{\frac{3}{10}}|00\rangle + \sqrt{\frac{3}{10}}|11\rangle + \sqrt{\frac{3}{10}}|22\rangle + \sqrt{\frac{1}{10}}|33\rangle$ is not possible deterministically via LOCC.