Assessed Problem 1: Purity of the outputs of random unitary channels

16 pts in total + 2 bonus points (part h).

Problem: Quantum information theorists love computing averages over random unitaries. To do so, we often make use of the following standard identities:

$$\int_{U(d)} U \otimes U^* \, dU = \frac{1}{d} |\operatorname{vec}(I_d)\rangle\langle \operatorname{vec}(I_d)| \tag{1}$$

and

$$\begin{aligned}&\int_{U(d)} U^{\otimes 2} \otimes \left(U^*\right)^{\otimes 2} dU \\ =&\frac{1}{d^2-1}\left(|\operatorname{vec}(I_{d^2})\rangle\langle \operatorname{vec}(I_{d^2})| - \frac{1}{d}|\operatorname{vec}(I_{d^2})\rangle\langle \operatorname{vec}(F)| - \frac{1}{d}|\operatorname{vec}(F)\rangle\langle \operatorname{vec}(I_{d^2})| + |\operatorname{vec}(F)\rangle\langle \operatorname{vec}(F)|\right)\end{aligned} \tag{2}$$

where d is the dimension of U, $U(d)$ is the uniform ensemble of all unitaries of dimension d, I_d is the identity of dimension d, and $F := \sum_{i,j=1}^{d} |i,j\rangle\langle j,i|$ is the SWAP operator.

Remark: You do not need to understand where these identities come from, or even fully understand them, to answer this question! (But if you've previously studied group averaging maybe you can see how they would be derived.)

This aim of this problem is for you to figure out how to use ***vectorization*** to use these helpful identities.

Let us start by considering the channel

$$\mathcal{E}_1(\rho) = \int_{U(d)} U\rho U^\dagger dU \,. \tag{3}$$

a) Use Eq. (1) to explicitly compute $\mathcal{E}_1(\rho) = \int_{U(d)} U\rho U^\dagger \, dU$. Can you state a set of Kraus operators to represent this channel?
2 pts. 1 pt for Haar integral. 1 pt for Kraus set.
Answer: Use the vectorization identity and Eq. (1), we find

$$\begin{aligned}|\operatorname{vec}(\mathcal{E}_1)\rangle &= \int_{U(d)} U \otimes U^* \, |\operatorname{vec}(\rho)\rangle dU &(4)\\ &= \int_{U(d)} U \otimes U^* dU \, |\operatorname{vec}(\rho)\rangle &(5)\\ &= \int dU \mathcal{A} = \frac{1}{d}|\operatorname{vec}(I_d)\rangle\langle \operatorname{vec}(I_d)|\operatorname{vec}(\rho)\rangle &(6)\\ &= \frac{1}{d}\operatorname{Tr}[\rho]|\operatorname{vec}(I_d)\rangle = \frac{1}{d}|\operatorname{vec}(I_d)\rangle \,. &(7)\end{aligned}$$

Hence the channel output is always the maximally mixed state:

$$\mathcal{E}_1(\rho) = \frac{I_d}{d} \,.$$

This is a completely depolarizing channel on a qudit. A valid Kraus set is

$$K_{ij} = \frac{1}{\sqrt{d}}|i\rangle\langle j|, \qquad i,j = 0, \cdots d-1 \,.$$

b) Compute the purity of $\mathcal{E}_1(\rho)$ assuming that the input ρ is a pure state.
1 pt
Answer: The purity for maximally mixed state is always $\frac{1}{d}$, regardless of the input state ρ.

c) Compare this to the average purity of a state $\rho_{\rm out}(U) = U\rho U^\dagger$ under the application of a random U. That is, compute

$$\int_{U(d)} \mathrm{Tr}[\rho_{\rm out}(U)^2]\, dU = \int_{U(d)} \mathrm{Tr}[(U\rho U^\dagger)^2]\, dU\,. \tag{8}$$

Comment briefly (a sentence or two will do) on why your answer here is different to your answer above (the answer is simple - do not confuse yourself looking for something more sophisticated answer).
2 pts. 1pt for finding the purity. 1pt for a good comment.
Answer: Note that now we are computing the purity averaged over random unitaries, instead of the purity of the averaged state. Since the unitary operation don't change the purity of states, for any U, $\mathrm{Tr}[(U\rho U^\dagger)^2] = \mathrm{Tr}[\rho^2]$. Hence

$$\int_{U(d)} \mathrm{Tr}[(U\rho U^\dagger)^2]\, dU = \int_{U(d)} dU \mathrm{Tr}[\rho^2] = \mathrm{Tr}[\rho^2]\,.$$

For completeness we give the direct calculation of the second moment integral using the SWAP operator. Let F be the SWAP on $\mathbb{C}^d \otimes \mathbb{C}^d$ and define $X := \rho \otimes \rho$. We have

$$\begin{aligned}\int_{U(d)} \mathrm{Tr}\big[(U\rho U^\dagger)^2\big]\, dU &= \int_{U(d)} \mathrm{Tr}\big(F\,(U\otimes U)X(U^\dagger \otimes U^\dagger)\big)\, dU \\ &= \langle\!\langle F|\,T\,|X\rangle\!\rangle,\end{aligned}$$

where T is given by Eq. (2)

$$T \;=\; \int_{U(d)} U^{\otimes 2} \otimes (U^*)^{\otimes 2}\, dU = \frac{1}{d^2-1}\Big(|I\rangle\!\rangle\langle\!\langle I| - \frac{1}{d}|I\rangle\!\rangle\langle\!\langle F| - \frac{1}{d}|F\rangle\!\rangle\langle\!\langle I| + |F\rangle\!\rangle\langle\!\langle F|\Big)\,.$$

Use the fact that the trace of product of matrices equals the inner product of their vectorization, we find

$$\langle\!\langle F|I\rangle\!\rangle = \mathrm{Tr}(F) = d, \quad \langle\!\langle F|F\rangle\!\rangle = \mathrm{Tr}(I) = d^2,$$

$$\langle\!\langle I|X\rangle\!\rangle = \mathrm{Tr}(X) = \mathrm{Tr}(\rho)^2 = 1, \quad \langle\!\langle F|X\rangle\!\rangle = \mathrm{Tr}(FX) = \mathrm{Tr}(\rho^2).$$

Then

$$\begin{aligned}\int_{U(d)} \mathrm{Tr}\big[(U\rho U^\dagger)^2\big]\, dU &= \frac{1}{d^2-1}\Big(d\cdot 1 - \tfrac{1}{d}\, d\cdot \mathrm{Tr}(\rho^2) - \tfrac{1}{d}\, d^2\cdot 1 + d^2\, \mathrm{Tr}(\rho^2)\Big) \\ &= \mathrm{Tr}(\rho^2)\,,\end{aligned}$$

which fits the fact that unitaries don't change the purity.
We thus see that the averaged purity do not equal the purity of the averaged state. This is to be expected because: i) the channel induces mixing! we start with a pure state and we end up with a mixture of pure states! but fundamentally each of those states is pure and so the average purity is pure!! (not sure why I have included so many exclamation marks here). ii) the purity operation $Tr[(\cdot)^2]$ is not linear.

Let's now consider a tripartite system consisting of three subspaces of dimension d. As shown in Fig. 1, we consider the quantity

$$\rho_{\rm out} = U_{3,2}(\rho_3 \otimes \sigma_2)U_{3,2}^\dagger \simeq U_{2,3}(\sigma_2 \otimes \rho_3)U_{2,3}^\dagger \tag{9}$$

where

$$\sigma_2 = \mathrm{Tr}_1[U_{1,2}(\rho_1 \otimes \rho_2)U_{1,2}^\dagger]\,, \tag{10}$$

and U's have dimension d^2, the density operators ρ's and σ have dimension d, and we specify the subspace on which they act in their subscripts.

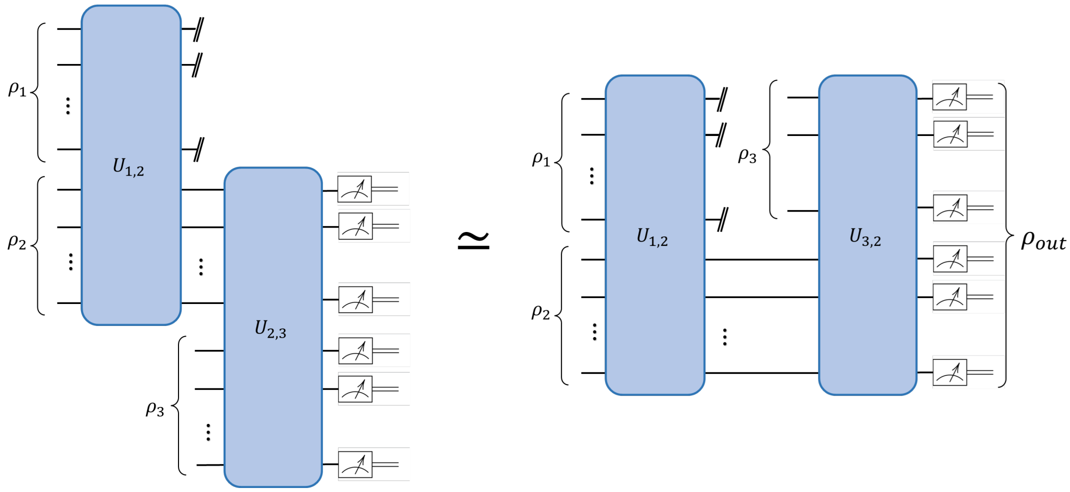

FIG. 1. Illustration of the tripartite system and the operations.

d) Let $U_{1,2}$ and $U_{3,2}$ be **independent** randomly chosen unitaries. That is

$$\mathcal{E}_2(\rho_2) = \int_{U(d^2)} U_{3,2}\big(\rho_3 \otimes \bar{\sigma}_2(\rho_2)\big)U_{3,2}^\dagger \, dU_{3,2} \tag{11}$$

where

$$\bar{\sigma}_2(\rho_2) = \int_{U(d^2)} \mathrm{Tr}_1[U_{1,2}(\rho_1 \otimes \rho_2)U_{1,2}^\dagger] dU_{1,2} \,, \tag{12}$$

Compute $\mathcal{E}_2(\rho_2)$ in this case. Can you find a set of Kraus operators to represent this channel?
1 pts. 0.5 for the channel. 0.5 for the Krauss
Answer: As $U_{1,2}$ is independent of $U_{3,2}$, $\bar{\sigma}_2(\rho_2)$ is also independent of $U_{3,2}$. Hence, we repeat the calculation in question a) but for the composite space, and find

$$\mathcal{E}_2(\rho_2) = \frac{I_d \otimes I_d}{d^2} \,,$$

with Kraus operators

$$K_{ij} = \frac{1}{d}|i\rangle\langle j|, \qquad i = 0, \cdots d^2 - 1 \,, j = 0, \cdots d - 1$$

where note that the input state is of dimension d whereas the output state is dimension d^2.

e) Now suppose $U_{1,2} = U_{3,2} = U$ are the **same** randomly drawn unitary, and we define the channel

$$\mathcal{E}_3(\rho_2) = \int_{U(d^2)} U(\rho_3 \otimes \mathrm{Tr}_1[U(\rho_1 \otimes \rho_2)U^\dagger])U^\dagger \, dU \,. \tag{13}$$

Compute the output of the channel $\mathcal{E}_2(\rho_2)$. Can you find a set of Kraus operators to represent this channel?
5 pts. 2 pt for the circuit decomposition into Haar integral form. 2 pt for Haar integral. 1 pt for the Kraus operator.
Answer: Define the state of integrand as

$$\rho_{out} := U(\rho_3 \otimes \mathrm{Tr}_1[U(\rho_1 \otimes \rho_2)U^\dagger])U^\dagger \,.$$

It can be rewrite as

$$\rho_{out} = \sum_{X \in \mathcal{B}(d^2)} \mathrm{Tr}\left[\mathrm{Tr}_1[U(\rho_1 \otimes \rho_2)U^\dagger]X^T\right] U(\rho_3 \otimes X)U^\dagger \tag{14}$$

$$= \sum_{X \in \mathcal{B}(d^2)} \mathrm{Tr}\left[(I_d \otimes X^T)U(\rho_1 \otimes \rho_2)U^\dagger\right] U(\rho_3 \otimes X)U^\dagger \tag{15}$$

Hence,

$$|\text{vec}(\rho_{out})\rangle = \sum_{X\in\mathcal{B}(d^2)} \langle \text{vec}(I_d \otimes X^T)|U\otimes U^*|\text{vec}(\rho_1\otimes\rho_2)\rangle\; U\otimes U^*|\text{vec}(\rho_3\otimes X)\rangle \tag{16}$$

$$= \sum_{X\in\mathcal{B}(d^2)} \Big(\langle \text{vec}(I_d\otimes X^T)|\otimes I_{d^2}^{\otimes 2}\Big)(U\otimes U^*)^{\otimes 2}\Big(|\text{vec}(\rho_1\otimes\rho_2)\rangle\otimes|\text{vec}(\rho_3\otimes X)\rangle\Big). \tag{17}$$

Recall the integral formula for second moment Eq. (2),

$$\int_{U(d^2)} U^{\otimes 2}\;\otimes\;\left(U^*\right)^{\otimes 2} dU$$
$$= \frac{1}{d^4-1}\left(|\text{vec}(I_{d^4})\rangle\langle\text{vec}(I_{d^4})| - \frac{1}{d^2}|\text{vec}(I_{d^4})\rangle\langle\text{vec}(F)| - \frac{1}{d^2}|\text{vec}(F)\rangle\langle\text{vec}(I_{d^4})| + |\text{vec}(F)\rangle\langle\text{vec}(F)|\right), \tag{18}$$

for which we can compute

$$\Big(\langle\text{vec}(I_d\otimes X^T)|\otimes I_{d^2}^{\otimes 2}\Big)|\text{vec}(I_{d^4})\rangle = d\,\text{Tr}[X]|\text{vec}(I_{d^2})\rangle$$

$$\Big(\langle\text{vec}(I_d\otimes X^T)|\otimes I_{d^2}^{\otimes 2}\Big)|\text{vec}(F)\rangle = |\text{vec}(I_d\otimes X^T)\rangle$$

$$\langle\text{vec}(I_{d^2})|\Big(|\text{vec}(\rho_1\otimes\rho_2)\rangle\otimes|\text{vec}(\rho_3\otimes X)\rangle\Big) = \text{Tr}[X]$$

$$\langle\text{vec}(F)|\Big(|\text{vec}(\rho_1\otimes\rho_2)\rangle\otimes|\text{vec}(\rho_3\otimes X)\rangle\Big) = \text{Tr}[\rho_1\rho_3]\text{Tr}[\rho_2 X]$$

Combining, we have

$$|\text{vec}(\mathcal{E}_3(\rho_2))\rangle = \int_{U(d^2)} dU\,\rho_{out} \tag{19}$$

$$= \frac{1}{d^4-1}\sum_X \Big(d\,\text{Tr}[X]\text{Tr}[X]|\text{vec}(I_{d^2})\rangle - \frac{1}{d^2}d\,\text{Tr}[X]\text{Tr}[\rho_1\rho_3]\text{Tr}[\rho_2 X]|\text{vec}(I_{d^2})\rangle \tag{20}$$

$$- \frac{1}{d^2}\text{Tr}[X]|\text{vec}(I_d\otimes X^T)\rangle + \text{Tr}[\rho_1\rho_3]\text{Tr}[\rho_2 X]|\text{vec}(I_d\otimes X^T)\rangle\Big) \tag{21}$$

$$= \frac{1}{d^4-1}\sum_X \Big(d\,\text{Tr}[X]^2|\text{vec}(I_{d^2})\rangle - \frac{1}{d}\text{Tr}[X]\text{Tr}[\rho_1\rho_3]\text{Tr}[\rho_2 X]|\text{vec}(I_{d^2})\rangle \tag{22}$$

$$- \frac{1}{d^2}\text{Tr}[X]|\text{vec}(I_d\otimes X^T)\rangle + \text{Tr}[\rho_1\rho_3]\text{Tr}[\rho_2 X]|\text{vec}(I_d\otimes X^T)\rangle\Big) \tag{23}$$

$$= \frac{1}{d^4-1}\Big(d^2|\text{vec}(I_{d^2})\rangle - \frac{1}{d}\text{Tr}[\rho_1\rho_3]|\text{vec}(I_{d^2})\rangle - \frac{1}{d^2}|\text{vec}(I_{d^2})\rangle + \text{Tr}[\rho_1\rho_3]|\text{vec}(I_d\otimes\rho_2)\rangle\Big) \tag{24}$$

$$= |\text{vec}(\frac{I_{d^2}}{d^2})\rangle + \frac{d}{d^4-1}\text{Tr}[\rho_1\rho_3]\Big(|\text{vec}(\frac{I_d}{d}\otimes\rho_2 - \frac{I_{d^2}}{d^2})\rangle\Big). \tag{25}$$

With that, we find the channel output

$$\mathcal{E}_3(\rho_2) = \frac{I_d}{d}\otimes\frac{I_d}{d} + \frac{d}{d^4-1}\text{Tr}[\rho_1\rho_3]\Big(\frac{I_d}{d}\otimes\rho_2 - \frac{I_d}{d}\otimes\frac{I_d}{d}\Big).$$

Now we find the Kraus operators for this channel. We first rewrite it as

$$\mathcal{E}_3(\rho) = \frac{I_d}{d}\otimes\frac{I_d}{d} + \frac{d}{d^4-1}\,\text{Tr}(\rho_1\rho_3)\left(\frac{I_d}{d}\otimes\rho - \frac{I_d}{d}\otimes\frac{I_d}{d}\right) = \frac{I_d}{d}\otimes\Big[(1-\alpha)\tfrac{I_d}{d} + \alpha\,\rho\Big],$$

with

$$\alpha := \frac{d}{d^4-1}\,\mathrm{Tr}(\rho_1\rho_3) \in [0,1].$$

We introduce the Heisenberg–Weyl unitaries on $\mathbb{C}^d$

$$X|j\rangle = |j+1 \bmod d\rangle, \qquad Z|j\rangle = \omega^j|j\rangle, \;\; \omega = e^{2\pi i/d}, \qquad W_{ab} = X^a Z^b,$$

and recall

$$\sum_{a,b} W_{ab} A W_{ab}^\dagger = d\;\mathrm{Tr}(A)\, I_d.$$

Define a depolarizing channel $\Lambda_\alpha(\rho) = (1-\alpha)\frac{I_d}{d} + \alpha\rho$ with Kraus operators

$$L_{00} = \sqrt{\alpha + \frac{1-\alpha}{d^2}}\, I_d, \qquad L_{ab} = \sqrt{\frac{1-\alpha}{d^2}}\, W_{ab} \quad \big((a,b)\neq(0,0)\big).$$

Then $\sum_{a,b} L_{ab}^\dagger L_{ab} = I_d$ and $\sum_{a,b} L_{ab}\rho L_{ab}^\dagger = (1-\alpha)\frac{I_d}{d} + \alpha\rho$.
A Kraus family for $\mathcal{E}_3 : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is

$$K_{i,(a,b)} = \frac{1}{\sqrt{d}}\big(|i\rangle \otimes L_{ab}\big), \qquad i = 0,\ldots,d-1, \;\; (a,b) \in \{0,\ldots,d-1\}^2,$$

so that

$$\sum_{i,a,b} K_{i,(a,b)}^\dagger K_{i,(a,b)} = \sum_{a,b} L_{ab}^\dagger L_{ab} = I_d,$$

and

$$\sum_{i,a,b} K_{i,(a,b)}\,\rho\, K_{i,(a,b)}^\dagger = \frac{I_d}{d} \otimes \sum_{a,b} L_{ab}\rho L_{ab}^\dagger = \frac{I_d}{d} \otimes \Big[(1-\alpha)\tfrac{I_d}{d} + \alpha\rho\Big] = \mathcal{E}_3(\rho).$$

f) Compute the purity of the output state for these two cases (Eq. (11) and Eq. (13)).

Comment briefly on how the purity differs between the two cases. Do you have an intuition as to why this is the case? (Again, just a sentence or two will do!)

(You may assume ρ_1, ρ_2 and ρ_3 are all pure).

2 pts. 0.5 pt for the first case (maximally mixed state). 1 pt for the second case. 0.5 pt for discussion.

Answer: The output state given by Eq. (11) from independent unitaries is the maximally mixed state of dimension d^2, which has the purity of $\frac{1}{d^2}$.

The output state from correlated dynamics in part e), Eq. (13), can be written as

$$\mathcal{E}_3(\rho_2) = p(\frac{I_d}{d} \otimes \rho_2) + (1-p)\frac{I_{d^2}}{d^2}\,,$$

where $p = \frac{d}{d^4-1}\mathrm{Tr}[\rho_1\rho_3]$. Hence the purity is

$$\begin{aligned}
\mathrm{Tr}[\mathcal{E}_3(\rho_2)^2] &= \mathrm{Tr}[p^2(\frac{I}{d^2} \otimes \rho_2^2) + (1-p)^2\frac{I_{d^2}}{d^4} + 2p(1-p)(\frac{I}{d^2} \otimes \frac{\rho_2}{d})] && (26)\\
&= p^2\frac{1}{d} + (1-p)^2\frac{1}{d^2} + 2p(1-p)\frac{1}{d^2} && (27)\\
&= p^2\frac{1}{d} + (1-p^2)\frac{1}{d^2}\,. && (28)
\end{aligned}$$

We see that the averaged state from correlated random dynamics (Eq.(13)) has potentially a purity greater than the one of the state (Eq. (11)) from independently random dynamics. It means that the independent one provides a stronger scrambling. Intuitively this makes sense because roughly purity quantifies how much classical randomness we have and things are more random when both unitaries are chosen randomly than when one is chosen randomly and used in both cases.

g) What changes if you instead compute the *average* purity of the random output states in both these cases. That is, the average purity of

3 pts. 1+1 pts for identifying both cases have the same purity (unitaries don't change the purity). 1 pt for the 2nd moment Haar integral.

- $\rho_{\text{out}}(U_{1,2}, U_{3,2}) = U_{3,2}(\rho_3 \otimes \sigma_2)U_{3,2}^\dagger$ with $\sigma_2 = \text{Tr}_1[U_{1,2}(\rho_1 \otimes \rho_2)U_{1,2}^\dagger]$, over $U_{1,2}$ and $U_{3,2}$ independently sampled from $U(d^2)$;

  Answer: Note that unitaries won't change the purity of states, so

$$\text{Tr}[\rho_{out}^2] = \text{Tr}[(\rho_3 \otimes \sigma_2)^2] \tag{29}$$
$$= \text{Tr}[\rho_3^2 \otimes \sigma_2^2)] \tag{30}$$
$$= \text{Tr}[\rho_3^2]\text{Tr}[\sigma_2^2]\,. \tag{31}$$

  Now we only need to find

$$\overline{\text{Tr}\,\sigma_2^2} = \int_{U(D)} \text{Tr}\Big[(\text{Tr}_1\, U\rho U^\dagger)^2\Big]\, dU = \int \text{Tr}\Big[(U\rho U^\dagger)^{\otimes 2}\,(I_{11'} \otimes F_{22'})\Big]\, dU,$$

  with $\rho = \rho_1 \otimes \rho_2$ and F the swap.
  Using $\langle\!\langle A|B\rangle\!\rangle = \text{Tr}(A^\dagger B)$ and $(U \otimes U^*)|\rho\rangle\!\rangle = |U\rho U^\dagger\rangle\!\rangle$,

$$\text{Tr}(\sigma_2^2) = \langle\!\langle U\rho U^\dagger \,|\, A \,|\, U\rho U^\dagger\rangle\!\rangle, \qquad A := |I_d\rangle\!\rangle\langle\!\langle I_d| \otimes I_{d^2}.$$

  Let $D = d^2$. The second-moment Haar formula gives the twirl

$$\mathcal{T}(A) = \int (U \otimes U^*)^\dagger A (U \otimes U^*)\, dU = \frac{1}{D^2 - 1}\Big[\Big(\text{Tr}A - \tfrac{1}{D}\text{Tr}(AF)\Big)I + \Big(\text{Tr}(AF) - \tfrac{1}{D}\text{Tr}A\Big)F\Big]\,, \tag{32}$$

  where F is the swap on $\mathbb{C}^D \otimes \mathbb{C}^D$. Hence

$$\mathbb{E}_U[\text{Tr}(\sigma_2^2)] = \langle\!\langle \rho \,|\, \mathcal{T}(A) \,|\, \rho\rangle\!\rangle.$$

  For $A = |I_d\rangle\!\rangle\langle\!\langle I_d| \otimes I_{d^2}$,

$$\text{Tr}A = d^3, \qquad \text{Tr}(AF) = d^2.$$

  Moreover,

$$\langle\!\langle \rho|I|\rho\rangle\!\rangle = \text{Tr}(\rho^2) = \text{Tr}(\rho_1^2)\text{Tr}(\rho_2^2), \qquad \langle\!\langle \rho|F|\rho\rangle\!\rangle = (\text{Tr}\rho)^2 = 1.$$

  Insert these into (32) to obtain

$$\mathbb{E}_U[\text{Tr}(\sigma_2^2)] = \frac{d\big(1 + \text{Tr}(\rho_1^2)\text{Tr}(\rho_2^2)\big)}{d^2 + 1}.$$

  If ρ_1, ρ_2 are pure, the value reduces to $2d/(d^2+1)$. If both are maximally mixed, it gives $1/d$.

- $\rho_{\text{out}}(U) = U(\rho_3 \otimes \text{Tr}_1[U(\rho_1 \otimes \rho_2)U^\dagger])U^\dagger$, over U sampled from $U(d^2)$.

  Answer: Similarly,

$$\text{Tr}[\rho_{out}^2] = \text{Tr}[(\rho_3 \otimes \text{Tr}_1[U(\rho_1 \otimes \rho_2)U^\dagger])^2] \tag{33}$$
$$= \text{Tr}[\rho_3^2]\text{Tr}[\text{Tr}_1[U(\rho_1 \otimes \rho_2)U^\dagger]^2]\,, \tag{34}$$

  meaning that the average purity is the same as the one in the previous case.

h) Bonus: I don't want you to do it, I'm not quite that cruel, but if I asked you to compute the average purity of $\rho_{\text{out}}(U) = \text{Tr}_3[U(\rho_3 \otimes \text{Tr}_1[U(\rho_1 \otimes \rho_2)U^\dagger])]U^\dagger]$ could you do it? If not, why not?
2 pts (max). 1 pt for arguing that it's analytically possible but hard due to 4! many terms. 1 pts for additional work, e.g. numerical simulation.
Answer: The provided formulas are not sufficient for this calculation, because here one needs to compute a 4th moment Haar integral. But in principle yes - results of Haar integral of any degree in U can be determined, see e.g. [Collins, Int. Math. Res. Not., (17):953-982, 2003.] for general procedures. However, the number of terms will increase dramatically – $t!$ terms for t-th moment integral, such that their exact values becomes intractable.
I've had to compute 4th moments and leading terms of arbitrary t-th moments in the past and I wouldn't inflict that on a master's student :)