Quantum Information and Quantum Computing, Solutions 9

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In this problem set we are going to deal with the generalisation of an important concept like the entanglement from pure states to density matrices.

Problem 1 : An entanglement witness for statistical mixtures

1. Let's consider a separable state on two subsystems described by the density matrix

$$\rho_s = \sum_k p_k \, \rho_k^{(1)} \otimes \rho_k^{(2)} \, , \tag{1}$$

and an observable $A_1 : \mathcal{H}_1 \to \mathcal{H}_1$ of subsystem 1. In this case its action on the whole subsystem is $A = A_1 \otimes \mathbb{I}_2$. For this reason we have

$$\begin{aligned} \left\langle \hat{A} \right\rangle &= \mathrm{Tr}\left(\hat{A}\hat{\rho}\right) && (2)\\ &= \sum_k p_k \mathrm{Tr}\left(\hat{A}\hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)}\right) && (3)\\ &= \sum_k p_k \mathrm{Tr}\left(\hat{A}_1\hat{\rho}_k^{(1)}\right)\mathrm{Tr}\left(\hat{\rho}_k^{(2)}\right) && (4)\\ &= \sum_k p_k \mathrm{Tr}\left(\hat{A}_1\hat{\rho}_k^{(1)}\right) && (5) \end{aligned}$$

since $\mathrm{Tr}\left(\hat{\rho}_k^{(2)}\right) = 1$ for a density matrix. This means that the expectation value of an observable acting only on a subsystem is independent from other subsystems if the density matrix is separable.

2. Formally the density matrix of the state $|\psi_{\mathrm{GHZ}}\rangle$ is

$$\begin{aligned} \hat{\rho}_{ABC} &= |\psi_{\mathrm{GHZ}}\rangle \langle\psi_{\mathrm{GHZ}}| && (6)\\ &= \frac{1}{2}\left(|000\rangle + |111\rangle\right)\left(\langle 000| + \langle 111|\right) && (7) \end{aligned}$$

that is a 8×8 matrix with $\frac{1}{2}$ at each corner of it.

The density matrix associated to the statistical mixture describing the subsystem formed by the qubits owned by Bob and Charlie can be obtained by making the partial trace over the

Alice subsystem:

$$\begin{aligned}
\hat{\rho}_{BC} &= \mathrm{Tr}_A\left[\rho_{ABC}\right] && (8)\\
&= \langle 0_A|\,\hat{\rho}_{ABC}\,|0_A\rangle + \langle 1_A|\,\hat{\rho}_{ABC}\,|1_A\rangle && (9)\\
&= \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) && (10)\\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} && (11)\\
&= \frac{1}{2}\left(\hat{\rho}_0^{(B)}\otimes\hat{\rho}_0^{(C)} + \hat{\rho}_1^{(B)}\otimes\hat{\rho}_1^{(C)}\right) && (12)
\end{aligned}$$

where $\hat{\rho}_0^{(j)} = |0_j\rangle\langle 0_j|$ and $\hat{\rho}_1^{(j)} = |1_j\rangle\langle 1_j|$. We notice that the last line has the same structure of Eq. 1, so it is separable.

3. For a separable density matrix $\hat{\rho}_s$ the partial transpose operation (wrt the second subsystem) acts like

$$\hat{\rho}_s^{T_B} = \sum_k \hat{\rho}_k^{(A)}\otimes\left(\hat{\rho}_k^{(B)}\right)^T \tag{13}$$

and the transpose matrices still have the properties of a density matrix, namely

$$\begin{aligned}
\mathrm{Tr}\left(\hat{\rho}_k^{(B)}\right)^T &= \mathrm{Tr}\left(\hat{\rho}_k^{(B)}\right) = 1 && (14)\\
\left(\left(\hat{\rho}_k^{(B)}\right)^T\right)^\dagger &= \left(\hat{\rho}_k^{(B)}\right)^T && (15)\\
\left(\hat{\rho}_k^{(B)}\right)^T &\text{ and } \hat{\rho}_k^{(B)} \text{ have the same proper values} && (16)
\end{aligned}$$

so $\hat{\rho}_s^{T_B}$ is still a separable density matrix.

4. As before, the density matrix for A, B, C and D has the form $\hat{\rho}_{ABCD} = |\psi_S\rangle\langle\psi_S|$. If we calculate the partial trace on A we have

$$\hat{\rho}_{BCD} = \langle 0_A| \hat{\rho}_{ABCD} |0_A\rangle + \langle 1_A| \hat{\rho}_{ABCD} |1_A\rangle \tag{17}$$

$$= \frac{1}{4}(|000\rangle\langle 000| + |000\rangle\langle 011| + |011\rangle\langle 000| + |011\rangle\langle 011|)$$
$$+ \frac{1}{4}(|100\rangle\langle 100| - |100\rangle\langle 111| - |111\rangle\langle 100| + |111\rangle\langle 111|) \tag{18}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

Now if we apply the partial transpose wrt to a single subsystem and we prove that it is no longer a density matrix, we have satisfied the sufficient condition for a density matrix to be entanlged. In order to do this, let's calculate the partial transpose wrt C,

$$\hat{\rho}_{BCD}^{T_C} = \frac{1}{4}(|000\rangle\langle 000| + |010\rangle\langle 001| + |001\rangle\langle 010| + |011\rangle\langle 011|) \tag{20}$$
$$+ \frac{1}{4}(|100\rangle\langle 100| - |110\rangle\langle 101| - |101\rangle\langle 110| + |111\rangle\langle 111|) \tag{21}$$

$$= = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

that is a matrix of which is easy to calculate the eigenvalues in diagonal blocks. From both the first and the second blocks that have non diagonal elements we have that the secular equation is $\lambda^2 - 1 = 0$ and gives two eigenvalues $\lambda = \pm 1$. For this reason, the partially-transposed matrix isn't positive semi-definite and it's sufficient to affirm that $\hat{\rho}_{BCD}$ describes an entangled state.

Problem 2 : Entanglement entropy

1. Since $\hat{\rho}$ is a diagonal density matrix, we have that

$$\ln(\hat{\rho}) = \begin{pmatrix} \ln(x) & 0 \\ 0 & \ln(1-x) \end{pmatrix} \tag{23}$$

So the von Neumann density has the form

$$\begin{aligned} S(x) &= -\mathrm{Tr}\left[\begin{pmatrix} x\ln(x) & 0 \\ 0 & (1-x)\ln(1-x) \end{pmatrix}\right] & (24) \\ &= -\left[x\ln(x) + (1-x)\ln(1-x)\right] & (25) \end{aligned}$$

As a first thing, we can notice that, since $\lim_{x\to 0} x\ln(x) = 0$, we have $S(0) = S(1) = 0$. Then we can calculate the derivative wrt x

$$\frac{dS(x)}{dx} = \ln\left(\frac{1-x}{x}\right) \tag{26}$$

and studying the sign we get

- if $x = 0.5 \to (1-x)/x = 1$ and $S'(x) = 0$
- if $x < 0.5 \to (1-x)/x > 1$ and $S'(x) > 0$
- if $x > 0.5 \to (1-x)/x < 1$ and $S'(x) < 0$

so $S(x)$ has a maximum in $x = \frac{1}{2}$, that is $S(\frac{1}{2}) = ln(2)$. We can note also, by substituting $1-x \to y$, that the function is symmetric wrt $x = \frac{1}{2}$. The plot of the von Neumann entropy can be seen in Fig. 1

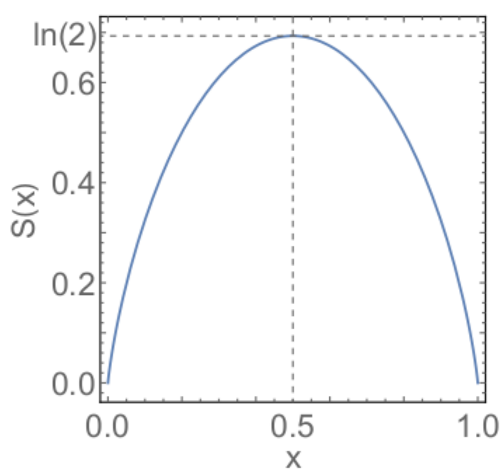

Figure 1: Von Neumann entropy as a function of the parameter x

To know for which value of x the state is pure, we have to calculate the purity $\mathrm{Tr}(\hat{\rho}^2) = x^2 + (1-x)^2$ and impose it equal to 1. This has solution $x = 0, 1$ and for these values we have $S_{\mathrm{pure}}(x) = S(0) = S(1) = 0$ as it should be. instead, when $x = \frac{1}{2}$, we have the maximum value of entropy, in this case, we have a statistical mixture of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ with equal weights. In this sense, we can interpret $S(x)$ as a measure of "how mixed" is the state given.

2. Given the state $|\psi\rangle$ and its Schmidt decomposition

$$|\psi\rangle = \sum_{j=1,2} \lambda_j |a_j\rangle \otimes |b_j\rangle, \quad \text{with } \lambda_j \in \mathbb{R}\,,\ 0 \leq \lambda_j \leq 1\,,\ \text{and} \sum_{j=1,2} \lambda_j^2 = 1\,. \tag{27}$$

we can compute the the reduced density matrices $\hat{\rho}_A$ and $\hat{\rho}_B$:

$$\hat{\rho}_A = \mathrm{Tr}_B\left[|\psi\rangle\langle\psi|\right] \tag{28}$$

$$= \sum_j \langle b_j|\psi\rangle\langle\psi|b_j\rangle \tag{29}$$

$$= \sum_{j,k} \lambda_k^2 |a_k\rangle\langle b_j|b_k\rangle\langle b_k|b_j\rangle\langle a_k| \tag{30}$$

$$= \sum_j \lambda_j^2 |a_j\rangle\langle a_j| \tag{31}$$

$$\hat{\rho}_B = \sum_j \lambda_j^2 |b_j\rangle\langle b_j| \tag{32}$$

and in their basis, respectively $\{|a_1\rangle, |a_2\rangle\}$ and $\{|b_1\rangle, |b_2\rangle\}$, we obtain

$$\hat{\rho}_A = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} = \hat{\rho}_B \tag{33}$$

3. From the expression above, we have $S(\hat{\rho}_A) = S(\hat{\rho}_B)$ since a trace operation is independent from the basis in which it is done. This makes sense, since the "measure of mixture" can't be different from subsystem to subsystem.

   - If $|\psi\rangle$ is a separable state, then the Schmidt decomposition is $|\psi\rangle = |a\rangle \otimes |b\rangle$, that is the case $\lambda_1 = 1$ and $\lambda_2 = 0$, from which $S(\hat{\rho}_A) = S(\hat{\rho}_B) = 0$ (recalling what we have seen in the first point).
   - if $|\psi\rangle$ is a maximally entangled state, then for a couple of basis we can write $|\psi\rangle = (|a_1\rangle \otimes |b_1\rangle + |a_2\rangle \otimes |b_2\rangle)/2$ and we compute $S(\hat{\rho}_A) = S(\hat{\rho}_B) = \ln(2) = S_{\max}$

   This means that given a maximally entangled state, it partial trace on a subsystem gives a maximally mixed density matrix. Since $S(\rho_A)$ measures "how mixed" is the state ρ_A , it can be considered also a measure of "how entangled" is the original state $\rho = |\psi\rangle\langle\psi|$