Quantum Information and Quantum Computing, Solutions 6

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In this problem set we explore the quantum phase estimation algorithm further.

Problem 1 : Quantum phase estimation

We have seen that, if the phase ϕ of the eigenvalue can be expressed exactly using t qubits, then we are able to retrieve its exact value by applying the quantum phase estimation algorithm

1. Now let's suppose $\phi = 0\underset{b}{(.)}\phi_1\phi_2\ldots\phi_t\phi_{t+1}\ldots\phi_s$, with $s > t$. Then, $2^t\phi = \phi_1\phi_2\ldots\phi_t\underset{b}{(.)}\phi_{t+1}\ldots\phi_s$ is not an integer and hence we are only able to retrieve $b = \phi_1\phi_2\ldots\phi_t$, its best integer approximation from below. That is, we can only recover the first t bits and lose the information about the $s-t$ bits beyond the (binary) decimal place.

   Considering the algorithm, after applying the Hadamard gates to the first register and the cascaded c-U^{2^k} gates to the second, we have that the first t qubits are in the state

$$|\Psi(\phi)\rangle_t = \frac{1}{2^{t/2}}\sum_{x=0}^{2^t-1} e^{2\pi i\phi x}|x\rangle_t. \tag{1}$$

   If we now apply the QFT† we get

$$\begin{aligned}\mathrm{QFT}^\dagger|\Psi(\phi)\rangle_t &= \frac{1}{2^t}\sum_{x=0}^{2^t-1} e^{2\pi i\phi x}\sum_{y=0}^{2^t-1} e^{-2\pi i\frac{xy}{2^t}}|y\rangle_t \qquad (2)\\ &= \frac{1}{2^t}\sum_{x=0}^{2^t-1}\sum_{y=0}^{2^t-1}\exp\left[-2\pi i x\frac{y-2^t\phi}{2^t}\right]|y\rangle_t \qquad (3)\end{aligned}$$

   if $y - 2^t\phi \neq 0\ \forall y$, the sum over x does not give $2^n\delta_{y,2^t\phi}$. It is rather expressed as the sum of the first 2^t elements of a geometric series (with x as exponent)

$$\begin{aligned}\mathrm{QFT}^\dagger|\Psi(\phi)\rangle_t &= \frac{1}{2^t}\sum_{y=0}^{2^t-1}\frac{1-\exp\left[-2\pi i(y-2^t\phi)\right]}{1-\exp\left[-2\pi i(y-2^t\phi)2^{-t}\right]}|y\rangle_t \qquad (4)\\ &= \sum_{y=0}^{2^t-1} f_y(\phi;t)|y\rangle_t. \qquad (5)\end{aligned}$$

   This is the expression of the state we were looking for: a superposition of all the possible outcomes, each with probability $p(y)$.

$$p(y) = |f_y(\phi;t)|^2 = \frac{1}{2^{2t}} \frac{1 - \cos\left[2\pi(y - 2^t\phi)\right]}{1 - \cos\left[2\pi(y - 2^t\phi)2^{-t}\right]} \tag{6}$$

2. Now, we want to compute $p(|m - b| > e)$ where b is again the best approximation and m is the result of the measurement on the first register.
   - To retrieve an upper bound, we have to rewrite the coefficients of the state calculated in the previous point in a more convenient form.
   - Then, we define $\delta = \phi - b2^{-t}$ as the error given by considering only a t-bit representation of the floating point number ϕ.
   - Given that $b = \lfloor \phi \rfloor$, it is natural to see that $0 \leq \delta < 2^{-t}$.
   - Then, we would like to rewrite the summation on all the possible states as a sum on all the possible errors $|l| = |m - b|$ we can make by measuring the bit string m.
   - We know that we can only measure bit strings representing numbers from 0 to $2^t - 1$ (that is the range of numbers we can represent using t bits) and that the expression $e^{2\pi i m/2^t}$ is periodic with period 2^t we have that $|l| \leq 2^t - 1$.

   Combining these points, we obtain,

$$\mathrm{QFT}^\dagger|\Psi(\phi)\rangle_t = \frac{1}{2^t} \sum_{l=-2^{t-1}}^{2^t-1} \frac{1 - \exp\left[-2\pi i(2^t\delta - l)\right]}{1 - \exp\left[-2\pi i(2^t\delta - l)2^{-t}\right]} |l \bmod 2^t\rangle_t. \tag{7}$$

   Now, let's recall two useful inequalities

$$|1 - e^{i\theta}| \leq 2 \qquad \forall\theta\,, \tag{8}$$

$$|1 - e^{i\theta}| \geq \frac{2|\theta|}{\pi} \qquad \text{if} -\pi \leq \theta \leq \pi\,. \tag{9}$$

   We note that when $-2^t - 1 < l \leq 2^t - 1$ we have $-\pi < 2\pi(\delta - l2^{-t}) \leq \pi$

   Then it follows

$$p(l) \leq \frac{1}{4} \frac{1}{(l - 2^t\delta)^2}\,. \tag{10}$$

   Considering e as the maximum possible error that we want to make on estimating ϕ we obtain

$$p(|m - b| > e) = \sum_{l=-2^{t-1}+1}^{-(e-1)} p(l) + \sum_{l=e+1}^{2^t-1} p(l) \tag{11}$$

$$\leq \frac{1}{4}\left[\sum_{l=-2^{t-1}+1}^{-(e-1)} \frac{1}{(l - 2^t\delta)^2} + \sum_{l=e+1}^{2^t-1} \frac{1}{(l - 2^t\delta)^2}\right] \tag{12}$$

Accounting for $0 < 2^t\delta < 1$, we can find an upper bound to the probability

$$p(|m-b| > e) \leq \frac{1}{4}\left[\sum_{l=-2^{t-1}+1}^{-(e-1)} \frac{1}{l^2} + \sum_{l=e+1}^{2^t-1} \frac{1}{(l-1)^2}\right] \tag{13}$$

$$\leq \frac{1}{2}\sum_{l=e}^{2^{t-1}-1} \frac{1}{l^2}, \tag{14}$$

The inequality still holds if we replace the sum with an integral

$$p(|m-b| > e) \leq \frac{1}{2}\int_{l=e-1}^{2^{t-1}-1} \frac{1}{l^2} \leq \frac{1}{2(e-1)}. \tag{15}$$

This upper bound now depends only on the accuracy that we want to achieve.

3. Now, if we require

$$|m-b| < 2^{t-n} - 1, \tag{16}$$

that is, if we are required to estimate $2^t\phi$ with an accuracy better than 2^{-n}, with $n < t$, with probability $p = 1-\epsilon$, then the previous result leads to

$$p(|m-b| < 2^{t-n} - 1) \geq 1 - \frac{1}{2(2^{t-n}-2)} \tag{17}$$

so

$$\epsilon = \frac{1}{2(2^{t-n}-2)} \tag{18}$$

therefore

$$t = n + \log_2\left(2 + \frac{1}{2\epsilon}\right). \tag{19}$$

This is the minimal value of t needed to achieve the required accuracy.

4. We now suppose that we know the eigenvalue ϕ_0. Our goal is to set the second register in a state that is a good estimate of the eigenvector whose eigenvalue is ϕ_0, namely $|\phi_0\rangle$. As before, assume that $2^t\phi$ can be expressed exactly as a t-bit register. If we prepare the second register in $|\psi_{in}\rangle$, after applying the Hadamard gates on the first register, we obtain

$$|\Psi_1\rangle = \frac{1}{2^{t/2}}\sum_{x=0}^{2^t-1}\sum_j c_j |x\rangle_t \otimes |\phi_j\rangle \tag{20}$$

where we have decomposed $|\psi_{in}\rangle = \sum_j c_j|\phi_j\rangle$ on the basis of the eigenvector $|\phi_j\rangle$ with eigenvalue ϕ_j. Now if we apply the controlled unitaries and the QFT† we get

$$|\Psi_2\rangle = \frac{1}{2^t}\sum_{x=0}^{2^t-1}\sum_{y=0}^{2^t-1}\sum_j c_j \mathrm{exp}\left[-2\pi i x\frac{y-2^t\phi_j}{2^t}\right]|y\rangle_t \otimes |\phi_j\rangle \tag{21}$$

since ϕ_j can be expressed exactly as a t- bit integer, the sum over x gives us the term $2^t\delta_{y,2^t\phi_j}$ and we get

$$|\Psi_2\rangle = \sum_j c_j|2^t\phi_j\rangle_t \otimes |\phi_j\rangle \tag{22}$$

so the probability of measuring $2^t\phi_0$, and therefore to have $|\phi_0\rangle$ in the second register, is exactly $|c_0|^2$. The higher is $|c_0|^2$, the higher the probability to get the state we want.

5. Now $2^t\phi_0$ can't be expressed exactly as t-bit integer; therefore, the sum over x after the phase estimation protocol does not give the δ but the expression $f_y(\phi_j;t)$ wrote down explicitly in the first point of the exercise.

   If we suppose that we measured $b = \lfloor 2^t\phi_0 \rfloor$, then, the final state of our system is

$$|\Psi_3\rangle = \sum_j c_j f_b(\phi_j,t)|b\rangle_t \otimes |\phi_j\rangle \tag{23}$$

   and the probability to obtain the state $|\phi_0\rangle$ in the second register is

$$p(|\phi_0\rangle) = \frac{|c_0|^2}{2^{2t}}\frac{1-\cos\left(2\pi\delta 2^t\right)}{1-\cos\left(2\pi\delta\right)}. \tag{24}$$

6. In both cases (whether $2^t\phi_0$ is an exact t-bit integer or not), the probability of getting the second register prepared in the state we are seeking is proportional to $|c_0|^2$. Considering that $|c_0|^2$ is the overlap between the initial state $|\Psi_{init}\rangle$ in which we prepare the second register and the eigenstate $|\phi_0\rangle$ we want to obtain, a crucial link for the success of the algorithm is to make an educated guess on the initial state, based on past or apriori knowledge on the system we are studying.