Quantum Information and Quantum Computing, Problem set 9

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The goal of this problem set is to familiarize with the notion of the density matrix and its properties, and to learn a couple of important tools such as the partial transpose and the entanglement entropy.

Problem 1 : An entanglement witness for statistical mixtures

A quantum system is composed of two subsystems and described in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the Hilbert spaces of the two subsystems. A state is separable if its density matrix can be expressed as

$$\rho_s = \sum_k p_k \, \rho_k^{(1)} \otimes \rho_k^{(2)} \,, \tag{1}$$

where $\sum_k p_k = 1$, $p_k \geq 0$, and $\rho_k^{(1)}$ and $\rho_k^{(2)}$ density matrices defined in the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. A state that cannot be expressed in the form (1) is an entangled state.

1. Prove that, for such a separable state, the expectation value of an observable A_1 of subsystem 1 doesn't depend on subsystem 2, i.e. it doesn't depend on $\rho_k^{(2)}$.

2. Three actors, Alice, Bob, and Charlie, each own one qubit. The overall system of three qubits is in the state $|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. The three actors are very far apart, and none of them can know the quantum state completely. Compute (in the computational basis) the density matrix associated to the statistical mixture describing the subsystem formed by the qubits owned by Bob and Charlie. Prove that the density matrix so obtained is separable.

We now introduce the operation called *partial transpose* (not to be confused with the partial trace). Consider a density matrix ρ describing a system which is composed of two subsystems. Let us denote by $\{|i\rangle, |j\rangle, \ldots\}$ the states of the basis of subsystem 1, by $\{|\mu\rangle, |\nu\rangle, \ldots\}$ those of subsystem 2, and by $\{|i\mu\rangle, |i\nu\rangle, |j\mu\rangle, |j\nu\rangle, \ldots\}$ those of the basis of the total system. If ρ has matrix elements $\rho_{i\mu,j\nu} = \langle i\mu|\rho|j\nu\rangle$, then the matrix elements of the density matrix ρ^{T_P}, obtained by applying the partial transpose with respect to subsystem 2, are defined as $(\rho^{T_P})_{i\mu,j\nu} = \langle i\nu|\rho|j\mu\rangle$. An analogous definition holds for the partial transpose with respect to subsystem 1.

3. Prove that, for a separable state of the two subsystems, the partial transpose $\rho_s^{T_P}$ with respect to either one of the two subsystems is still a valid density matrix – i.e. it is hermitian, positive semi-definite, and has trace 1.
Notice that in this way we will have proven a <u>*necessary*</u> *condition for a density matrix to be separable. This automatically implies a* <u>*sufficient*</u> *condition for a density matrix to be non-separable, namely entangled.*

4. Four actors, Alice, Bob, Charlie and David, each own a qubit. The system of the four qubits is in the state $|\psi_S\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$. Like before, the actors are very

far apart. Compute the density matrix associated to the state of the subsystem formed by the three qubits of Bob, Charlie and David. Prove that the statistical mixture obtained is an entangled state, using the partial transpose criterion that we have just derived.
Hint: Study the eigenvalues of the partial transpose.

Problem 2 : Entanglement entropy

The state of a qubit is described by the following density matrix

$$\rho = \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix} \qquad 0 \leq x \leq 1 \tag{2}$$

1. Compute the von Neumann entropy of ρ, defined as $S(\rho) = -\mathrm{Tr}(\rho \ln(\rho)) = S(x)$. Study the function $S(x)$ in the range where it is defined. What's the value of $S(x)$ for a pure state? For which value of x does $S(x)$ reach a maximum, and what is the corresponding state?

Let us now consider two qubits. We will adopt here the tool known as *Schmidt decomposition*: it can be proven that it is always possible to find two orthonormal bases $\{|a_1\rangle, |a_2\rangle\}$ and $\{|b_1\rangle, |b_2\rangle\}$ which allow expressing an arbitrary pure state of the two qubits as

$$|\psi\rangle = \sum_{j=1,2} \lambda_j |a_j\rangle \otimes |b_j\rangle, \quad \text{where } \lambda_j \in \mathbb{R}\,,\ 0 \leq \lambda_j \leq 1\,,\ \text{and } \sum_{j=1,2} \lambda_j^2 = 1\,. \tag{3}$$

2. Using the Schmidt decomposition, compute the reduced density matrices ρ_A and ρ_B of each of the two qubits, assuming that the total system is in the state $|\psi\rangle$.

3. Compute now the von Neumann entropies $S(\rho_A)$ and $S(\rho_B)$. How are these two values related to each other? What is the value of $S(\rho_A)$ if $|\psi\rangle$ is a separable state? And if $|\psi\rangle$ is a maximally entangled state of the two systems? On the basis of your findings, explain why $S(\rho_A)$ is called *entanglement entropy.*