# Statistical Physics of Computation 2025 - Exercises

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## Week 9

This week we compute the spectrum of random matrices using the replica method. In particular, given a certain symmetric random matrix $M \in \mathbb{R}^{n\times n}$, we are interested in determining the probability density $\rho_M(\lambda)$ of the eigenvalues $\lambda_1, \ldots \lambda_n$

$$\rho_M(\lambda) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta(\lambda - \lambda_i) \tag{1}$$

Throughout the exercise we will assume that $\rho_M(\lambda)$ will concentrate to a bounded distribution, and that will be no issue in exchanging the limit $n \to \infty$ with any other one. For convenience we will denote with $\mathrm{Tr[M]}$ the trace of $M$, and with $\mathrm{tr[M]} = \mathrm{Tr[M]}/n$ the trace divided by the dimension.

### 9.1 Setting up a replica computation for the spectrum

We start by identifying the Stieltjes transform as the central object we want to compute, and detailing its relation to a replica computation

1. Define the Stieltjes transform $g_M(z)$ of the matrix $M$ as the trace of the resolvent

$$g_M(z) = \mathrm{tr} \frac{1}{z\mathbb{I} - M}\,, \tag{2}$$

where by $1/M$ we mean the matrix inverse of $M$. We also recall this representation of the delta function given by the Sokhotski–Plemelj theorem

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon\to 0^+} \mathrm{Im}\, \frac{1}{x - i\epsilon}\,, \tag{3}$$

where Im denotes the imaginary part of the function. Obtain the following relation between $g_M(z)$ and $\rho_M(\lambda)$

$$\rho_M(\lambda) = \frac{1}{\pi} \lim_{\epsilon\to 0^+} \mathrm{Im}\, g_M(\lambda - i\epsilon)\,. \tag{4}$$

Recall the identity

$$\mathrm{Tr}[f(M)] = \sum_{i=1}^{n} f(\lambda_i)\,, \tag{5}$$

which gives us

$$g_M(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{z - \lambda_i}\,. \tag{6}$$

Now we manipulate the density using the identity provided

$$\rho_M(\lambda) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \frac{1}{n} \sum_{i=1}^{n} \text{Im} \frac{1}{\lambda - \lambda_i - i\epsilon} , \tag{7}$$

where we exchanged the two limits. We then notice that

$$\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda - \lambda_i - i\epsilon} = \text{tr} \frac{1}{(\lambda - i\epsilon)\mathbb{I} - M} = g_M(\lambda - i\epsilon) . \tag{8}$$

This can then be substituted in, giving the result.

2. The previous point shows that knowing $g_M(z)$ is enough to obtain the spectrum, now we show that $g_M(z)$ is related to the determinant of the resolvent. Derive the expression

$$\log \det(z\mathbb{I} - M) = \sum_{i=1}^{n} \log(z - \lambda_i) \tag{9}$$

and use it to write

$$g_M(z) = n^{-1} \partial_z \log \det(z\mathbb{I} - M) \tag{10}$$

We start with

$$\det(z - M) = \prod_{i=1}^{n} (z - \lambda_i) . \tag{11}$$

Applying a logarithm to both sides we get

$$\log \det(z - M) = \sum_{i=1}^{n} \log(z - \lambda_i) . \tag{12}$$

Taking a derivative with respect to $z$

$$\partial_z \log \det(z - M) = \sum_{i=1}^{n} \frac{1}{z - \lambda_i} . \tag{13}$$

The result is obtained by dividing both sides by $n$.

3. Turns out that the determinant is a quantity we can compute using the replica method. Argue why we can write

$$\det(z\mathbb{I} - M)^{-\frac{1}{2}} = \int e^{-\frac{z\|w\|^2}{2} + \frac{1}{2} \sum_{i,j=1}^{n} M_{ij} w_i w_j} \, \mathrm{d}w \tag{14}$$

where $w$ is a $n$-dimensional vector, and $w_i$ are its components. Show that one can write the average Stieltjes transform as

$$\mathbb{E}_M[g_M(z)] = -\frac{2}{n} \partial_z \lim_{r \to 0} \frac{\mathbb{E}_M[\mathcal{Z}^r] - 1}{r} . \tag{15}$$

What is $\mathcal{Z}$? Notice that contrary to Bayes-optimal computations, here there is no hidden signal, hence we have $r$ replicas with $r \to 0$ and not $r + 1$.

The first identity is immediate: it's simply a Gaussian integral! When computing the logarithm of the determinant we are actually just computing a free entropy. For the second identity we have

$$\mathbb{E}_M[g_M(z)] = n^{-1}\partial_z \mathbb{E}_M[\log\det(z-M)] \tag{16}$$

In the identity from before we have the power $-1/2$ of the determinant, so we multiply and divide the right hand side to have

$$\mathbb{E}_M[g_M(z)] = -\frac{2}{n}\partial_z \mathbb{E}_M[\log\det(z-M)^{-\frac{1}{2}}]\,. \tag{17}$$

The remaining difficulty is in computing the average of the log, but that is precisely what the replica method is for. We call $\mathcal{Z} = \det(z-M)^{-\frac{1}{2}}$, then

$$\mathbb{E}_M[\log \mathcal{Z}] = \lim_{r\to 0} \frac{\mathbb{E}_M[\mathcal{Z}^r]-1}{r}\,, \tag{18}$$

which gives the solution.

## 9.2 The replica method for the Wigner model

In this part we go through the replica method, computing the spectrum for the simplest class of matrices: the Gaussian Orthogonal Ensemble (GOE). Specifically, it's the ensemble of symmetric matrices with $M = \frac{1}{\sqrt{2n}}(G+G^\top)$, with $G_{ij} \sim \mathcal{N}(0,1)$. Apart from an overall scaling, and a difference in the diagonal that does not matter for $n \gg 1$, this is the noise matrix of the spiked-Wigner model.

We want to compute the moments of the partition function $\mathcal{Z}$

$$\mathcal{Z} = \int e^{\frac{z\|w\|^2}{2} - \frac{1}{2}\sum_{i,j=1}^n M_{ij}w_iw_j}\,\mathrm{d}w\,. \tag{19}$$

Notice this is slightly different (overall sign in the exponent) from what you got in the previous point. We will clarify this later.

1. Write the $r$-th moment of $\mathcal{Z}$ as

$$\mathbb{E}_M[\mathcal{Z}^r] = \int e^{\frac{z}{2}\sum_{a=1}^r \|w^a\|^2} \prod_{i,j}^n \mathbb{E}_{g\sim N(0,1)}\left[e^{-\frac{1}{\sqrt{2n}}\sum_{a=1}^r g w_i^a w_j^a}\right] \mathrm{d}w^a \tag{20}$$

where by $\mathrm{d}w^a$ we mean that we integrate over all the $r$ replicas.

First we replicate

$$\mathbb{E}_M[\mathcal{Z}^r] = \int e^{\frac{z}{2}\sum_{a=1}^r \|w^a\|^2 + \frac{1}{2}\sum_{a=1}^r \sum_{i,j=1}^n M_{ij}w_i^a w_j^a}\,\mathrm{d}w^a\,. \tag{21}$$

Then, we notice that the average over $M$ must be taken only over the second piece in the exponent

$$\mathbb{E}_M[\mathcal{Z}^r] = \int e^{\frac{z}{2}\sum_{a=1}^r \|w^a\|^2}\mathbb{E}_M\left[e^{-\frac{1}{2}\sum_{a=1}^r \sum_{i,j=1}^n M_{ij}w_i^a w_j^a}\right]\mathrm{d}w^a\,. \tag{22}$$

Looking at the term in the exponential we can notice that

$$\frac{1}{2}\sum_{i,j=1}^n M_{ij}w_i^a w_j^a = \frac{1}{2\sqrt{2n}}\sum_{i,j=1}^n (G_{ij}+G_{ji})w_i^a w_j^a = \frac{1}{\sqrt{2n}}\sum_{i,j=1}^n G_{ij}w_i^a w_j^a\,, \tag{23}$$

Since the elements of $G$ are independent we have

$$\mathbb{E}_G\left[e^{-\frac{1}{\sqrt{2n}}\sum_{a=1}^r\sum_{i,j=1}^n G_{ij}w_i^a w_j^a}\right]=\prod_{i,j}^n \mathbb{E}_{g\sim N(0,1)}\left[e^{-\frac{1}{\sqrt{2n}}\sum_{a=1}^r g w_i^a w_j^a}\right] \tag{24}$$

obtaining the result.

2. Show that

$$\prod_{i,j=1}^n \mathbb{E}_{g\sim N(0,1)}\left[e^{-\frac{1}{\sqrt{2n}}\sum_{a=1}^r g w_i^a w_j^a}\right]=e^{\frac{n}{4}\sum_{a,b=1}^r\left[\sum_{i=1}^n\frac{w_i^a w_i^b}{n}\right]^2} \tag{25}$$

The expectation is a Gaussian integral

$$\prod_{i,j=1}^n \mathbb{E}_G\left[e^{-\frac{1}{\sqrt{2n}}\sum_{a=1}^r G_{ij}w_i^a w_j^a}\right]=\prod_{i,j=1}^n e^{\frac{1}{4n}\sum_{a,b=1}^r w_i^a w_j^a w_i^b w_j^b}\,, \tag{26}$$

where in squaring the exponent we introduced an extra index $b$. Then

$$\prod_{i,j=1}^n e^{\frac{1}{4n}\sum_{a,b=1}^r w_i^a w_j^a w_i^b w_j^b}=e^{\frac{1}{4n}\sum_{a,b=1}^r\sum_{i=1}^n w_i^a w_i^b\sum_{j=1}^n w_j^a w_j^b}\,, \tag{27}$$

where we permuted the $w$ and put the sum back in the exponent. Noticing that

$$\sum_{i=1}^n w_i^a w_i^b\sum_{j=1}^n w_j^a w_j^b=\left[\sum_{i=1}^n w_i^a w_i^b\right]^2 \tag{28}$$

gives the result.

3. Let's recap for a moment. At this point of the computation you should agree that we have this expression for the moment of the partition function

$$\mathbb{E}_M[\mathcal{Z}^r]=\int e^{\frac{zn}{2}\sum_{a=1}^r\frac{\|w^a\|^2}{n}+\frac{n}{4}\sum_{a,b=1}^r\left[\sum_{i=1}^n\frac{w_i^a w_i^b}{n}\right]^2}\,\mathrm{d}w^a\,. \tag{29}$$

It's the moment to introduce our order parameter $q_{ab}$

$$q_{ab}=\frac{w_i^a w_i^b}{n}\,. \tag{30}$$

Rewrite the moment as

$$\mathbb{E}_M[\mathcal{Z}^r]=\int e^{\frac{zn}{2}\sum_{a=1}^r q_{aa}+\frac{n}{4}\sum_{a,b=1}^r q_{ab}^2+I_{\text{entropy}}(q)}\,\mathrm{d}q_{ab}\,. \tag{31}$$

Here we use the convention $\mathrm{d}q_{ab}$ to indicate the integral over $\{dq_{ab}\}_{1\le a\le b\le r}$. How is $I_{\text{entropy}}(q)$ defined?

Introducing the order parameter with a delta function gives us

$$\mathbb{E}_M[\mathcal{Z}^r]=\int e^{\frac{zn}{2}\sum_{a=1}^r q_{aa}+\frac{n}{4}\sum_{a,b=1}^r q_{ab}^2}\prod_{a\le b}^r\int\delta\left(nq_{ab}-\sum_{i=1}^n w_i^a w_i^b\right)\mathrm{d}w^a\,\mathrm{d}q_{ab}\,. \tag{32}$$

Introducing the order parameters make a new "entropic" piece of the free entropy emerge

$$I_{\text{entropy}}(q) = \log \left[ \int \prod_{a \le b}^{r} \delta \left( n q_{ab} - \sum_{i=1}^{n} w_i^a w_i^b \right) \mathrm{d}w^a \right] . \tag{33}$$

The name comes from the fact that this piece "counts" the number of configurations of the variables $w^a$ such that the overlap will be $q_{ab}$.

4. For the following, you can assume that

$$I_{\text{entropy}}(q) = \frac{n}{2} \log \det q \, . \tag{34}$$

You will derive this in a later part of the exercise. Make sure you agree that

$$\mathbb{E}_M[\mathcal{Z}^r] = \int e^{\frac{zn}{2} \operatorname{Tr} q + \frac{n}{4} \operatorname{Tr} q^2 + \frac{n}{2} \log \det q} \, \mathrm{d}q_{ab} \, . \tag{35}$$

Since $q$ is symmetric we can decompose it into a diagonal matrix of eigenvalues $\Lambda$ and a rotation matrix $O$ as $q = O^\top \Lambda O$. Show that the argument of the integral only depends on $\Lambda$. Then, show that

$$\mathbb{E}_M[\mathcal{Z}^r] = \int e^{\frac{n}{2} \sum_{a=1}^{r} \left[ z\Lambda_a + \frac{\Lambda_a^2}{2} + \log \Lambda_a \right] + \sum_{a<b} \log |\Lambda_a - \Lambda_b|} \, \mathrm{d}\Lambda \, , \tag{36}$$

using (and not deriving) Weyl's integration formula

$$\mathrm{d}q = \mu(O) \prod_{a<b} |\Lambda_a - \Lambda_b| \, \mathrm{d}\Lambda \, \mathrm{d}O \, , \tag{37}$$

where $\mathrm{d}O$ means integration over $n \times n$ rotation matrices $O$, and $\mu(O)$ is the uniform measure over rotation matrices.

The trace of a matrix is the sum of the eigenvalues, the trace of the square is sum of the square of the eigenvalues, the determinant is the product of the eigenvalues. All of these are independent of the eigenvector matrix $O$. This means that the integrand is independent of $O$, so one can preform the integration over it and remove from the expression.

5. This is almost an expression where all eigenvalues $\Lambda$ are independent, except for the term that came from the change of variables. It's reasonable to drop it in a first approximation for $n \gg 1$. Let's compute the integral by the saddle-point method. Show that the saddle point equation for each $\Lambda_a$ equals

$$-z = \Lambda_a + \frac{1}{\Lambda_a} \tag{38}$$

For each $a$ we have

$$\partial_{\Lambda_a} \left[ z\Lambda_a + \frac{\Lambda_a}{2} + \log \Lambda_a \right] = 0 \, , \tag{39}$$

which gives

$$z + \Lambda_a + \frac{1}{\Lambda_a} = 0 \, . \tag{40}$$

Rearranging gives the solution.

6. The equation above is not a bad approximation. It is in fact exact in the large $n$ limit. Show that without dropping the logarithm term the saddle point equation is for each $a$

$$z + \Lambda_a + \frac{1}{\Lambda_a} + \frac{1}{n}\sum_{b\neq a}\frac{1}{\Lambda_a - \Lambda_b} = 0\,. \tag{41}$$

Now let's stare at this equation for a moment. If the last piece is negligible (as we want to claim) it means that all the $\Lambda_a$ will be identical. This will however make the last fraction diverge, so the answer can't be that simple! In fact, what we should look at is an ansatz of this form

$$\Lambda_a = \Lambda^* + \frac{\epsilon_a}{\sqrt{n}} \tag{42}$$

where $\Lambda^*$ is a solution of the saddle point equation without the extra piece. What we are saying here is that at finite $n$ the $\Lambda_a$ will be different, and in the limit $n \to \infty$ they will have a typical distance of the order $1/\sqrt{n}$. Show that the ansatz of the scaling $1/\sqrt{n}$ is consistent, and derive (but do not solve) the equation satisfied by the corrections $\epsilon_a$. Finally, argue that the correction term is truly subleading.

The saddle point equation is obtained by simply taking the derivative. Notice the extra factor $1/n$ for the last piece. We now substitute our ansatz. We get

$$\begin{aligned} z + \Lambda_a^* + \frac{\epsilon_a}{\sqrt{n}} + \frac{1}{\Lambda_a^* + \frac{\epsilon_a}{\sqrt{n}}} + \frac{1}{\sqrt{n}}\sum_{b\neq a}\frac{1}{\epsilon_a - \epsilon_b} \approx \\ z + \Lambda_a^* + \frac{1}{\Lambda_a^*} + \frac{\epsilon_a}{\sqrt{n}}\left[1 - \frac{1}{(\Lambda_a^*)^2}\right] + \frac{1}{\sqrt{n}}\sum_{b\neq a}\frac{1}{\epsilon_a - \epsilon_b} = 0\,. \end{aligned} \tag{43}$$

The first three pieces sum to zero by definiiton of $\Lambda^*$, so we are left with

$$\epsilon_a = \frac{(\Lambda_a^*)^2}{1 - (\Lambda_a^*)^2}\sum_{b\neq a}\frac{1}{\epsilon_a - \epsilon_b}\,. \tag{44}$$

This verifies that the scaling is correct, otherwise we would have found factors $n$ in the equation, and is the equation for the corrections. It can be a fun mathematical challenge (outside of the scope of this exercise) to show that $\epsilon_a$ that solve this equation are the zeros of a Hermite polynomial. Finally, we notice that under this ansatz, the correction term is of order $1/n \times 1/(1/\sqrt{n}) = 1/\sqrt{n}$, confirming that it is subleading.

7. We now want to connect this replica computation with (15). Argue that the $\mathcal{Z}$ in (19) can for our purposes replace the $\det(z - M)^{-\frac{1}{2}}$ defined in (14).

One can be mapped in the other by the substitution $w \to iw$. This is allowed, as $w$ is just an integration variable. This change of variable is typically called Wick rotation, as it consists of a rotation of the integration domain from the real to the imaginary axis or vice-versa. One needs to be careful that under this integration domain deformation one does not cross any non-analyticity of the integrand. In this case, the integrand is the exponential of a quadratic form, thus is analytic and has no poles. As we This change would produce an extra constant, that will have no bearing in (15)

8. Show that $\mathbb{E}_M[g_M(z)]$ (for convenience just $g$ from now onward) is a solution of the equation

$$z = g + \frac{1}{g} \tag{45}$$

Using the saddle point method we get

$$\mathbb{E}_M[g_M(z)] = -\frac{2}{n}\partial_z \lim_{r\to 0}\frac{\mathbb{E}_M[\mathcal{Z}^r]-1}{r} = -\partial_z\left[z\bar{\Lambda}+\frac{\bar{\Lambda}^2}{2}+\log\bar{\Lambda}\right] = -\bar{\Lambda}\,, \tag{46}$$

where $\bar{\Lambda}$ is the solution of the saddle point equation in the question. Then one uses the saddle-point condition

$$\partial_{\bar{\Lambda}}\left[z\bar{\Lambda}+\frac{\bar{\Lambda}^2}{2}+\log\bar{\Lambda}\right] = 0\,. \tag{47}$$

9. We are at the end of the computation. Solve $g(z)$ in terms of $z$. To choose the right solution look at the definition of $g(z)$ for very large $|z|$.

   We recall the definition of $g$ and expand it

$$g(z) = \mathrm{tr}\frac{1}{z\mathbb{I}-M} = \sum_{k=0}^{\infty}\frac{\mathrm{tr}\,M^k}{z^{k+1}}\,, \tag{48}$$

   so at large $|z|$ one should have $g(z)\approx 1/z$. Solving for $g$ one gets

$$g(z) = \frac{z\pm\sqrt{z^2-4}}{2}\,. \tag{49}$$

   We can expand this at large $|z|$

$$\frac{z\pm\sqrt{z^2-4}}{2} = \frac{z}{2}\left(1\pm\sqrt{1-\frac{4}{z^2}}\right) = \frac{z}{2}\left(1\pm 1-\frac{2}{z^2}+\mathcal{O}(z^{-2})\right)\,. \tag{50}$$

   One has to thus choose the negative sign, giving

$$g(z) = \frac{z-\sqrt{z^2-4}}{2}\,. \tag{51}$$

10. Obtain finally the spectral distribution $\rho_M(\lambda)$

$$\rho_M(\lambda) = \mathbf{1}_{|\lambda|<2}\frac{\sqrt{4-\lambda^2}}{2\pi} \tag{52}$$

   where $\mathbf{1}_{|\lambda|<2}$ is the indicator function. Notice that the distribution is a semicircle of radius 2. This should be familiar to you from exercise 4.

   Let's write $g(\lambda-i\epsilon)$ for small $\epsilon$

$$g(\lambda-i\epsilon) = \frac{\lambda-i\epsilon+\sqrt{(\lambda-i\epsilon)^2-4}}{2} \approx \frac{\lambda+\sqrt{\lambda^2-2i\epsilon\lambda-4}}{2} \tag{53}$$

   This expression contains the solution but it's not very transparent. Im $\lambda$ is zero, so we can drop it. The square root is the important piece: if $|\lambda|>2$ then in the limit $\epsilon\to 0$ the square root will be real, so the density will be zero. On the other hand, if $|\lambda|<2$ then in the limit the square root will be pure imaginary, so we can write

$$\mathrm{Im}\,g(\lambda-i\epsilon) \approx \frac{\mathrm{Im}\,[i\sqrt{4-\lambda^2+2i\epsilon\lambda}]}{2} = \frac{\sqrt{4-\lambda^2}}{2}\,. \tag{54}$$

   We are finally done!

### 9.3 Computation of entropic part

Here we compute the entropic piece

$$I_{\text{entropy}}(q) = \log \left[ \int \prod_{a\leq b}^{r} \delta \left( n q_{ab} - \sum_{i=1}^{n} w_i^a w_i^b \right) \, \mathrm{d}w^a \right] . \tag{55}$$

We will disregard all the $2\pi$ factors, as they will be completely irrelevant in the result.

1. Introduce the Lagrange multiplier matrix $\hat{q}$ to rewrite the integral in the log as

$$I_{\text{entropy}}(q) = \int \mathrm{d}\hat{q} \, \exp \left\{ n \sum_{a\leq b}^{r} \hat{q}_{ab} q_{ab} \right\} \prod_{i=1}^{n} \int \exp \left\{ - \sum_{a\leq b}^{r} \hat{q}_{ab} w_i^a w_i^b \right\} \mathrm{d}w^a \tag{56}$$

One transforms the delta into an exponent and then integrates over the weights

$$\int \prod_{a\leq b}^{r} \delta \left( n q_{ab} - \sum_{i=1}^{n} w_i^a w_i^b \right) \, \mathrm{d}w^a = \tag{57}$$

$$\int \mathrm{d}\hat{q} \, \exp \left\{ n \sum_{a\leq b}^{r} \hat{q}_{ab} q_{ab} \right\} \prod_{i=1}^{n} \int \exp \left\{ - \sum_{a\leq b}^{r} \hat{q}_{ab} w_i^a w_i^b \right\} \mathrm{d}w^a \tag{58}$$

2. What we have above is almost a Gaussian integral, except that in the exponent the sums run over $a \leq b$. Show that one can write

$$I_{\text{entropy}}(q) = \int \mathrm{d}\hat{q} \, \exp \left\{ \frac{n}{2} \sum_{a,b}^{r} \bar{q}_{ab} q_{ab} \right\} \prod_{i=1}^{n} \int \exp \left\{ -\frac{1}{2} \sum_{a,b}^{r} \bar{q}_{ab} w_i^a w_i^b \right\} \mathrm{d}w^a \tag{59}$$

where $\bar{q}_{ab} = (1 + \delta_{ab}) \hat{q}_{ab}$.

Notice the following manipulation

$$\begin{aligned} -\sum_{a\leq b}^{r} \hat{q}_{ab} w_i^a w_i^b &= -\sum_{a=1}^{r} \hat{q}_{aa} (w_i^a)^2 - \sum_{a<b}^{r} \hat{q}_{ab} w_i^a w_i^b \\ &= -\sum_{a=1}^{r} \hat{q}_{aa} (w_i^a)^2 - \sum_{a\neq b}^{r} \frac{\hat{q}_{ab}}{2} w_i^a w_i^b \\ &= -\sum_{a,b=1}^{r} \frac{(1+\delta_{ab})\hat{q}_{ab}}{2} w_i^a w_i^b \, . \end{aligned} \tag{60}$$

The same can be done for the piece $\hat{q}_{ab} q_{ab}$.

3. Perform the integral over the weights to have

$$I_{\text{entropy}}(q) = \int \mathrm{d}\hat{q} \, \exp \left\{ \frac{n}{2} \sum_{a,b}^{r} \bar{q}_{ab} q_{ab} - \frac{n}{2} \log \det \bar{q} \right\} \tag{61}$$

The integral over $w^a$ decouples for each $w_i^a$, and each of them gives

$$\int \exp\left\{-\frac{1}{2}\sum_{a,b}^{r}\bar{q}_{ab}w_i^a w_i^b\right\}\mathrm{d}w_i^a = (\det\bar{q})^{-\frac{1}{2}}\,. \tag{62}$$

In total one gets

$$\prod_{i=1}^{n}\int \exp\left\{-\frac{1}{2}\sum_{a,b}^{r}\bar{q}_{ab}w_i^a w_i^b\right\}\mathrm{d}w^a = \exp\left\{-\frac{n}{2}\log\det\bar{q}\right\} \tag{63}$$

4. Solve the remaining integral using a saddle point method, obtaining the result

$$I_{\text{entropy}}(q) = \frac{n}{2}\log\det q \tag{64}$$

up to constant additive terms. It is useful to know that for any matrix $M$, then $\partial_{M_{ij}}[\log\det M] = [M^{-1}]_{ij}$.
The saddle point equation is

$$\partial_{\bar{q}}\left[\sum_{a,b}^{r}\bar{q}_{ab}q_{ab} - \log\det(\bar{q})\right] = 0\,, \tag{65}$$

or equivalently $\bar{q} = q^{-1}$ using the relation provided. We now have to plug this back in the exponent. For the first piece we get

$$\sum_{a,b}^{r}\bar{q}_{ab}q_{ab} = \mathrm{Tr}[\bar{q}\,q] = \mathrm{Tr}[\mathbb{I}]\,, \tag{66}$$

but this one is just a constant so it can be dropped.