# Statistical Physics of Computation 2025 - Exercises

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## Week 6

### 6.1 Phase diagrams of spiked-Wigner model

In this exercise we will take the result of the replica computation we previously did for the low-rank denoising problem and study the associated state equations in a couple of simple cases. Our starting point is the state equation on $m$:

$$m = \int Dz \mathrm{d}x_0 P_0\left(x_0\right) \frac{\int P_0\left(x\right) x\, x_0 \, \mathrm{d}x \;\; \exp\left\{-\frac{\lambda m}{2}x^2 + \sqrt{\lambda m}xz + \lambda m x x_0\right\}}{\int P_0\left(x\right) \mathrm{d}x \;\; \exp\left\{-\frac{\lambda m}{2}x^2 + \sqrt{\lambda m}xz + \lambda m x x_0\right\}} \tag{1}$$

which is the extremisation equation of the free entropy $\phi$

$$\begin{aligned}\phi &= \max_m \phi(m) \\ &= \max_m \left[-\frac{\lambda}{4}m^2 + \int Dz P_X\left(x_0\right)\mathrm{d}x_0 \log\left(\int P_X\left(x\right)\mathrm{d}x \; e^{-\frac{\lambda m}{2}x^2 + \sqrt{\lambda m}xz + \lambda m x x_0}\right)\right]\end{aligned} \tag{2}$$

where $Dz$ denotes integration against a standard Gaussian variable.

**Gaussian prior.** We start with a simple case, i.e. Gaussian prior on the components of $x$:

$$x, x_0 \sim \mathcal{N}(0,1)\,. \tag{3}$$

1. Show that the state equation becomes

$$m = \frac{\lambda m}{\lambda m + 1}\,. \tag{4}$$

2. Compute the variational free entropy $\phi(m)$ for this prior.
3. Solve the state equation. In principle it should have two solutions. How should one choose the right one? Use this criterion to find the correct $m(\lambda)$ for each $\lambda$.
4. Is there a phase transition? Of which order? Justify.

**Sparse binary prior.** A more interesting case is the sparse binary prior. The elements of $x$ have a probability $0 < \rho < 1$ of being one and $1 - \rho$ of being zero

$$P_0(x_i) = \rho\delta(x_i - 1) + (1-\rho)\delta(x_i) \quad \forall i\,, \tag{5}$$

where $\delta$ is the Dirac's delta.

5. Show that the state equation becomes

$$m = \int Dz \frac{\rho^2}{\rho + (1-\rho)e^{-\frac{\lambda m}{2} + z\sqrt{\lambda m}}} \tag{6}$$

6. We would like to solve the state equation, but it would only be possible to do so numerically. We do that by a fixed point iteration scheme

$$m^{t+1} = \int Dz \frac{\rho^2}{\rho + (1-\rho)e^{-\frac{\lambda m^t}{2} + z\sqrt{\lambda m^t}}} \,. \tag{7}$$

   Iterate numerically (in the language of your choice) the state evolution until convergence for a large value of $\rho$ (for example $\rho = 0.8$) for several value of $\lambda$. Use $m^{t=0} = 0.01$ and $m^{t=0} = 0.99$ as initializations. What do you observe? Is the behavior of $m$ at convergence dependent on the initialization? Are there any phase transitions?
   Hint: you will need to integrate (possibly by quadrature) over $z$. We advise to choose a suitable integration interval (something like $[-5, 5]$ would do).

7. We want to repeat the same numerical experiment for small $\rho$. First, it is useful to estimate the magnitude of the SNR and of the magnetization in that limit. Show that

$$Q = Q_* = \mathbb{E}_{x \sim P_0} x^2 = \rho \,. \tag{8}$$

   Argue that this implies $m < \rho$, so that it is useful to plot $m/\rho$ as our order parameter for small $\rho$. Argue that the strength of the signal scales linearly with $\rho$, hence the effective signal to noise ratio of the model is $\lambda\rho^2$.

8. Show numerically that for $\rho = 0.02$ small enough there is a first order phase transition by plotting $m$ at convergence as a function of $\lambda$. Hint: In order to see anything from your plot you should use the scaling in $m$ and $\lambda$ from the previous point. Remember to initialize the state equation both in $m^{t=0} = 0.01$ and $m^{t=0} = 0.99$.

As discussed in class, to find the precise point at which the first order phase transition happens one needs to compare the value of the free entropy at the two solutions involved in the transition. For this prior, this is not so simple to do, as computing numerically the free entropy for small $\rho$ leads to numerical instabilities. The solution is to expand the free entropy for small $\rho$ after rescaling the magnetization and the SNR to highlight their natural dependence on $\rho$, in order to obtain a numerically more manageable approximate expression. This whole procedure starts being outside of the scope of these lectures, so we do not cover it here.