Statistical Physics of Computation 2025 - Exercises

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Week 5

5.1 Replica computation for the spiked-Wigner model

We are given as observation a $N \times N$ symmetric matrix Y created as

$$Y = \sqrt{\frac{\lambda}{N}} \underbrace{x^* x^{*\intercal}}_{N\times N \text{ rank-one matrix}} + \underbrace{\xi}_{\text{symmetric iid noise}}$$

where $x^* \in \mathbb{R}^N$ with $x_i^* \overset{\text{i.i.d.}}{\sim} P_X(x)$, $\xi_{ij} = \xi_{ji} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ for $i \le j$. We keep the prior P_X generic, as long as it is factorized over all the components of the vector x.

Our task shall be to recover the vector x from the knowledge of Y, the signal-to-noise ratio λ and the prior P_X. As we saw during the lecture, this can be achieved using the posterior estimation, i.e. by computing the posterior distribution and evaluating some statistics over it.

In this exercise, you will compute the normalization factor of the posterior distribution, i.e. the partition function of the problem, and derive the state equation for the overlap order parameter.

1. Show that the posterior distribution $P(x|Y)$ for the problem can be written as

$$P(x|\mathbf{Y}) = \frac{1}{Z(\mathbf{Y})} \left[\prod_{i=1}^N P_X(x_i)\right] \left[\prod_{i\le j} \frac{e^{-\frac{\lambda}{2N} x_i^2 x_j^2 + \sqrt{\frac{\lambda}{N}} x_i x_j y_{ij}}}{\sqrt{2\pi}}\right] \tag{1}$$

for a specific $Z(\mathbf{Y})$. How is $Z(\mathbf{Y})$ defined for this measure?

One needs to recognize that the output channel distribution satisfies, for all $i \le j$, $P_{\text{out}}(y_{ij}|x) = N(y_{ij}, \sqrt{\lambda/N} x_i x_j, 1)$, so that

$$P(x|\mathbf{Y}) = \frac{1}{Z(\mathbf{Y})} \left[\prod_{i=1}^N P_X(x_i)\right] \left[\prod_{i\le j} \frac{e^{-\frac{1}{2}\left(y_{ij} - \sqrt{\frac{\lambda}{N}} x_i x_j\right)^2}}{\sqrt{2\pi}}\right] \tag{2}$$

where we notice in particular that the product in the Gaussian factor runs over $i \le j$ due to the symmetry of the problem. It's convenient to reabsorb all pieces independent of x from the exponential in the normalisation, allowing us to write

$$P(x|\mathbf{Y}) = \frac{1}{Z(\mathbf{Y})} \left[\prod_{i=1}^N P_X(x_i)\right] \left[\prod_{i\le j} \frac{e^{-\frac{\lambda}{2N} x_i^2 x_j^2 + \sqrt{\frac{\lambda}{N}} x_i x_j y_{ij}}}{\sqrt{2\pi}}\right] \tag{3}$$

where

$$Z(\mathbf{Y}) = \int \mathrm{d}x \left[\prod_{i=1}^{N} P_X(x_i)\right] \left[\prod_{i\leq j} \frac{e^{-\frac{\lambda}{2N}x_i^2x_j^2+\sqrt{\frac{\lambda}{N}}x_ix_jy_{ij}}}{\sqrt{2\pi}}\right] \tag{4}$$

We are interested in computing the averaged free entropy associated to the posterior distribution, i.e.

$$\lim_{N\to\infty} \mathbb{E}_Y \left[\frac{1}{N}\log Z(Y)\right]$$

which we can compute using the replica method by computing first

$$\mathbb{E}_Y Z(Y)^n\,. \tag{5}$$

2. Show that the averaged replicated partition function equals

$$\mathbb{E}_Y[Z^n] = \int \mathrm{d}Y e^{-\frac{1}{2}\sum_{i\leq j}y_{ij}^2} \prod_{\alpha=0}^{n} \int \mathrm{d}x^{(\alpha)} \left(\prod_{i=1}^{N} P_X\left(x_i^{(\alpha)}\right)\right) \left(\prod_{i\leq j} \frac{e^{-\frac{\lambda}{2N}\left(x_i^{(\alpha)}\right)^2\left(x_j^{(\alpha)}\right)^2+\sqrt{\frac{\lambda}{N}}x_i^{(\alpha)}x_j^{(\alpha)}y_{ij}}}{\sqrt{2\pi}}\right) \tag{6}$$

where you should notice that we are taking the product $n+1$ replicas.

We use the identity

$$\mathbb{E}_Y[F[Y]] = \int \mathrm{d}Y P(Y)F[Y] = \int \mathrm{d}Y \int \mathrm{d}x^* P_{\text{out}}(Y|x^*)P_X(x^*)F[Y] = \int \mathrm{d}Y Z[Y]e^{-\frac{1}{2}\sum_{i\leq j}y_{ij}^2}F[Y] \tag{7}$$

where the last step crucially depends on the fact that we reabsorbed all x-independent terms of the posterior in the definition of the partition function. Then, one has

$$\mathbb{E}_Y[Z[Y]^n] = \int \mathrm{d}Y e^{-\frac{1}{2}\sum_{i\leq j}y_{ij}^2} Z[Y]^{n+1} \tag{8}$$

and one gets the result by plugging in the definition of the partition function.

3. Integrate over the disorder, i.e. the observation Y, to get at leading order in N

$$\mathbb{E}_Y[Z^n] = \int \prod_{\alpha,i} P_X\left(x_i^{(\alpha)}\right)\mathrm{d}x_i^{(\alpha)}\;\exp\left(\frac{\lambda N}{2}\sum_{\alpha<\beta}\left(\sum_i \frac{x_i^{(\alpha)}x_i^{(\beta)}}{N}\right)^2\right) \tag{9}$$

$$
\begin{aligned}
\mathbb{E}_{\mathbf{Y}}\left[Z^n\right] = \int \mathrm{d}\mathbf{Y} \prod_{\alpha=0}^{n} \int \mathrm{d}x^{(\alpha)} \prod_i P_X\left(x_i^{(\alpha)}\right) (2\pi)^{-\frac{N(N+1)}{2}} \\
\exp\left(\sum_{\alpha=0}^{n}\sum_{i\leq j}\left[-\frac{y_{ij}^2}{2} - \frac{\lambda}{2N} {x_i^{(\alpha)}}^2 {x_j^{(\alpha)}}^2 + \sqrt{\frac{\lambda}{N}} y_{ij} x_i^{(\alpha)} x_j^{(\alpha)}\right]\right) \\
= \int \prod_{\alpha,i} P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \; \exp\left(\sum_{i\leq j}\left[-\frac{\lambda}{2N}\sum_\alpha {x_i^{(\alpha)}}^2 {x_j^{(\alpha)}}^2\right]\right) \underbrace{\prod_{i\leq j}\int \mathrm{d}y_{ij}\, \frac{e^{-\frac{y_{ij}^2}{2} + y_{ij}\left(\sqrt{\frac{\lambda}{N}}\sum_\alpha x_i^{(\alpha)} x_j^{(\alpha)}\right)}}{\sqrt{2\pi}}}_{\stackrel{(a)}{=} \exp\left(\frac{\lambda}{2N}\sum_{i\leq j}\sum_{\alpha,\beta} x_i^{(\alpha)} x_j^{(\alpha)} x_i^{(\beta)} x_j^{(\beta)}\right)} \\
\stackrel{(b)}{=} \int \prod_{\alpha,i} P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \; \exp\left(-\frac{\lambda N}{4}\sum_\alpha\left(\sum_i \frac{{x_i^{(\alpha)}}^2}{N}\right)^2 + \frac{\lambda N}{4}\sum_{\alpha,\beta}\left(\sum_i \frac{x_i^{(\alpha)} x_i^{(\beta)}}{N}\right)^2\right) \\
= \int \prod_{\alpha,i} P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \; \exp\left(\frac{\lambda N}{2}\sum_{\alpha<\beta}\left(\sum_i \frac{x_i^{(\alpha)} x_i^{(\beta)}}{N}\right)^2\right)
\end{aligned}
$$

(a) uses the fact that $\int \mathcal{D}z\; e^{az} = e^{a^2/2}$

(b) uses the fact that

$$
\sum_{i\leq j} \frac{a_i}{N}\frac{a_j}{N} = \frac{1}{2}\left(\sum_i \frac{a_i}{N}\right)^2 + \frac{1}{2}\sum_i \frac{a_i^2}{N^2}
$$

and neglect the second term since it scales like $\mathrm{O}(N^{-1})$.

4. Introduce the order parameter

$$
q_{\alpha\beta} = \frac{1}{N}\sum_i x_i^\alpha x_i^\beta \tag{10}
$$

using a Dirac delta function, and its conjugate $\hat{q}$ through the Fourier representation of the Dirac delta, and obtain

$$
\mathbb{E}_Y[Z^n] = \int \prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta}\, \mathrm{d}q_{\alpha\beta} \; \exp\left(N I_{\text{energy}}(q_{\alpha\beta}, \hat{q}_{\alpha\beta}) + N I_{\text{entropy}}(\hat{q}_{\alpha\beta})\right) \tag{11}
$$

where we defined

$$
I_{\text{energy}}(q_{\alpha\beta}, \hat{q}_{\alpha\beta}) = \frac{\lambda}{2}\sum_{\alpha<\beta} q_{\alpha\beta}^2 - \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta} \tag{12}
$$

and

$$
I_{\text{entropy}}(\hat{q}_{\alpha\beta}) = \log\left(\int \prod_\alpha P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \; \exp\left\{\sum_{\alpha<\beta} \hat{q}_{\alpha\beta} x_\alpha x_\beta\right\}\right) \tag{13}
$$

where we stress that here the integral over dx_α runs over the real numbers for all $\alpha = 0, \ldots, n$.

$$
\begin{aligned}
\mathbb{E}_{\mathbf{Y}}\left[Z^n\right] &\overset{(a)}{=} \int \prod_{\alpha,i} P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \int \prod_{\alpha<\beta} \delta\left(N q_{\alpha\beta} - \sum_i x_i^{(\alpha)} x_i^{(\beta)}\right) \mathrm{d}q_{\alpha\beta}\ \exp\left(\frac{\lambda N}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2\right) \\
&\overset{(b)}{=} \int \prod_{\alpha,i} P_X\left(x_i^{(\alpha)}\right) \mathrm{d}x_i^{(\alpha)} \int \prod_{\alpha<\beta} e^{-\hat{q}_{\alpha\beta} N q_{\alpha\beta} + \hat{q}_{\alpha\beta} \sum_i x_i^{(\alpha)} x_i^{(\beta)}} \mathrm{d}\hat{q}_{\alpha\beta}\, \mathrm{d}q_{\alpha\beta}\ \exp\left(\frac{\lambda N}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2\right) \\
&\overset{(c)}{=} \int \prod_{\alpha<\beta} \mathrm{d}\hat{q}_{\alpha\beta}\, \mathrm{d}q_{\alpha\beta}\ \exp\left(\frac{\lambda N}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 - N \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta}\right) \\
&\qquad\qquad \left\{\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha\ \exp\left(\sum_{\alpha<\beta} \hat{q}_{\alpha\beta} x_\alpha x_\beta\right)\right\}^N
\end{aligned}
$$

(a) partitions the huge integral according to overlap between two distinct replicas $q_{\alpha\beta}$ with definitions

$$
q_{\alpha\beta} = \frac{1}{N} \sum_i x_i^{(\alpha)} x_i^{(\beta)}, \quad \forall\, \alpha < \beta
$$

(b) introduces Fourier representation of the Dirac's delta

(c) change the order of integral and expectation. Moreover, $x_i^{(\alpha)}$ are iid for each i, the tuple $x_i^{(0)}, x_i^{(1)}, \cdots, x_i^{(n)}$ are identical distributed, so we switch to subscript notation $x_{(0)}, x_{(1)}, \cdots, x_{(n)}$ to get rid of component index i but keep the replica index α. This leads to the power N around the curly brackets.

5. Show that in the RS ansatz $q_{\alpha\beta} = q$, $\hat{q}_{\alpha\beta} = \hat{q}$, the energetic term satisfies

$$
\lim_{n\to 0} \frac{1}{n} I_{\text{energy}}(q_{\alpha\beta}, \hat{q}_{\alpha\beta}) = \frac{\lambda}{4} q^2 - \frac{q\hat{q}}{2} \tag{14}
$$

for q and $\hat{q}$ real numbers.

We need to expand I_{energy} for small n. Recall that $q_{\alpha\beta}$ is a square matrix of size $n+1$.

$$
\begin{aligned}
I_{\text{energy}}(q_{\alpha\beta}, \hat{q}_{\alpha\beta}) =& \frac{\lambda}{2} \sum_{\alpha<\beta} q_{\alpha\beta}^2 + \sum_{\alpha<\beta} q_{\alpha\beta}\hat{q}_{\alpha\beta} \\
=& \frac{\lambda}{2} \frac{(n+1)n}{2} q^2 - \frac{(n+1)n}{2} q\hat{q} \\
\approx& \frac{\lambda n}{4} q^2 - \frac{n q\hat{q}}{2}
\end{aligned}
$$

6. We now need to compute the entropic term I_{entropy} in the RS ansatz. This is more tricky. We start by proving an identity that will come useful later. Show that for any positive constant a we have the following identity:

$$
\exp\left\{\frac{ax^2}{2}\right\} \propto \int \exp\left\{-\frac{z^2}{2a} + zx\right\} \mathrm{d}z \tag{15}
$$

Typically, we call this the Hubbard-Stratonovich (HS) transformation.

One can integrate over z the right hand side recognizing that it is a Gaussian integral to obtain the result.

7. The HS transformation is particularly useful in spin glasses, as we can use it to "decouple" strongly interacting systems at the price of introducing a random external field z. We can see it in practice in this example. Show that

$$\exp\left\{\frac{1}{2}\sum_{i,j}^{N} x_i x_j\right\} \propto \int \prod_{i=1}^{N} \exp\left\{-\frac{z^2}{2} + z x_i\right\} \mathrm{dz} \tag{16}$$

Here by decoupling we mean that the left hand side is not a product over i independent terms (not factorized), as for all pairs i, j the variables x_i and x_j interact (if we imagine the exponent to be an energy function). The right hand side instead is factorized over i, modulo paying the price of a Gaussian integral. **(Bonus)** Try to use the HS transform to re-derive the Curie-Weiss model partition function without using any Dirac delta function.

The key observation is that

$$\exp\left\{\frac{1}{2}\sum_{i,j}^{N} x_i x_j\right\} = \exp\left\{\frac{1}{2}\left(\sum_{i=1}^{N} x_i\right)^2\right\} \tag{17}$$

One can then directly use HS.

8. Show that in the RS ansatz $q_{\alpha\beta} = q$, $\hat{q}_{\alpha\beta} = \hat{q}$, the entropic term satisfies

$$I_{\text{entropy}}(\hat{q}_{\alpha\beta}) = \log\left(\int Dz \left[\left(\int P_X(x)\,\mathrm{d}x\ \exp\left\{-\frac{\hat{q}}{2}x^2 + \sqrt{\hat{q}}xz\right\}\right)^{n+1}\right]\right) \tag{18}$$

for $\hat{q}$ real numbers. In the derivation, you will have to use the RS anstaz followed by the HS transform.

We have

$$
\begin{aligned}
I_{\text{entropy}}(\hat{q}_{\alpha\beta}) &= \log\left(\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \;\; \exp\left\{-\sum_{\alpha<\beta} \hat{q}_{\alpha\beta} x_\alpha x_\beta\right\}\right) \\
&= \log\left(\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \;\; \exp\left\{-\hat{q} \sum_{\alpha<\beta} x_\alpha x_\beta\right\}\right) \\
&= \log\left(\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \;\; \exp\left\{-\frac{\hat{q}}{2} \sum_{\alpha\neq\beta} x_\alpha x_\beta\right\}\right) \\
&= \log\left(\int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \;\; \exp\left\{-\frac{\hat{q}}{2} \sum_{\alpha} x_\alpha^2 + \frac{\hat{q}}{2}\left(\sum_{\alpha} x_\alpha\right)^2\right\}\right) \\
&= \log\left(\int \mathcal{D}z \int \prod_{\alpha} P_X\left(x_\alpha\right) \mathrm{d}x_\alpha \;\; \exp\left\{\sum_{\alpha}\left(-\frac{\hat{q}}{2} x_\alpha^2 + \sqrt{\hat{q}} x_\alpha z\right)\right\}\right) \\
&= \log\left(\int \mathcal{D}z \left(\int P_X\left(x\right) \mathrm{d}x \;\; \exp\left\{-\frac{\hat{q}}{2} x^2 + \sqrt{\hat{q}} x z\right\}\right)^{n+1}\right)
\end{aligned}
$$

where in the fifth passage we used the Hubbard-Stratonovich transformation (see previous replica computation).

9. Show that in the small n limit, the entropic term satisfies

$$
\lim_{n\to 0} \frac{1}{n} I_{\text{entropy}}(\hat{q}_{\alpha\beta}) = \int Dz\, I(\hat{q}, z) \log\left(I(\hat{q}, z)\right) \tag{19}
$$

where we defined $I(\hat{q}, z)$ as

$$
I(\hat{q}, z) = \int P_X\left(x\right) \mathrm{d}x \;\; \exp\left\{-\frac{\hat{q}}{2} x^2 + \sqrt{\hat{q}} x z\right\} \tag{20}
$$

We will need the following identity: for a random quantity X with mean one, in the small n limit we have

$$
\mathbb{E}[X^{n+1}] \sim \exp(n\mathbb{E}[X\log(X)]) \tag{21}
$$

$$
\begin{aligned}
\mathbb{E}[X^{n+1}] = \mathbb{E}\left[X e^{n\log(X)}\right] &\sim \mathbb{E}[X + nX\log(X)] = \mathbb{E}[X] + n\mathbb{E}[X\log(X)] = 1 + n\mathbb{E}[X\log(X)] \\
&= \exp(\log(1 + n\mathbb{E}[X\log(X)])) \sim \exp(n\mathbb{E}[X\log(X)])
\end{aligned}
$$

Now we notice that

$$
\int \mathcal{D}z I(\hat{q}, z) = \int \mathcal{D}z \int P_X\left(x\right) \mathrm{d}x \;\; \exp\left\{-\frac{\hat{q}}{2} x^2 + \sqrt{\hat{q}} x z\right\} = \int P_X\left(x\right) \mathrm{d}x = 1
$$

which gives us the result.

10. Show that the previous expression can be rewritten as

$$\lim_{n\to 0}\frac{1}{n}I_{\text{entropy}}(\hat{q}_{\alpha\beta}) = \int Dz \int P_X(x_0)\,\mathrm{d}x_0 \log\left(\int P_X(x)\,\mathrm{d}x\ \exp\left\{-\frac{\hat{q}}{2}x^2+\sqrt{\hat{q}}xz+\hat{q}xx_0\right\}\right) \tag{22}$$

One needs to do a bit of manipulations. First, we notice that

$$\exp\left\{-\frac{1}{2}z^2\right\} I(\hat{q},z) = \int P_X(x)\,\mathrm{d}x \exp\left\{-\frac{1}{2}\left(\sqrt{\hat{q}}x-z\right)^2\right\} \tag{23}$$

Finally, after a change of variable, we have

$$z \to t = z - \sqrt{\hat{q}}x_0 \tag{24}$$

where we indicate with x_0 the integration variable inside the first $I(\hat{q},z)$ (the one outside of the logarithm). This gives us

$$\int Dz\, I(\hat{q},z)\log\left(I(\hat{q},z)\right) = \int Dt \int P_X(x_0)\,\mathrm{d}x_0\, I(\hat{q},z)\log\left(I(\hat{q},t+\sqrt{\hat{q}}x_0)\right) \tag{25}$$

which is exactly the result we want.

11. Argue finally that the free entropy

$$\phi = \lim_{N\to\infty}\mathbb{E}_Y\left[\frac{1}{N}\log Z(Y)\right] \tag{26}$$

can be expressed as

$$\phi = \text{extr}_{q,\hat{q}}\left[\frac{\lambda}{4}q^2-\frac{q\hat{q}}{2}+\int Dz\int P_X(x_0)\,\mathrm{d}x_0\log\left(\int P_X(x)\,\mathrm{d}x\ \exp\left\{-\frac{\hat{q}}{2}x^2+\sqrt{\hat{q}}xz+\hat{q}xx_0\right\}\right)\right] \tag{27}$$

or equivalently as

$$\phi = \text{extr}_{q}\left[-\frac{\lambda}{4}q^2+\int Dz P_X(x_0)\,\mathrm{d}x_0\log\left(\int P_X(x)\,\mathrm{d}x\ \exp\left\{-\frac{\lambda q}{2}x^2+\sqrt{\lambda q}xz+\lambda qxx_0\right\}\right)\right] \tag{28}$$

The first expression just combines the previous results and uses the replica trick

$$\phi \approx \frac{\mathbb{E}Z(Y)^n-1}{Nn}\,. \tag{29}$$

The second expression comes from taking the fixed point on q, which gives $\hat{q}=\lambda q$, and substituting it back in.

12. **(Bonus)** Show that the state equations satisfies

$$q = \int Dz P_X(x_0)\frac{\int P_X(x)\,x\,x_0\,\mathrm{d}x\ \exp\left\{-\frac{\lambda q}{2}x^2+\sqrt{\lambda q}xz+\lambda qxx_0\right\}}{\int P_X(x)\,\mathrm{d}x\ \exp\left\{-\frac{\lambda q}{2}x^2+\sqrt{\lambda q}xz+\lambda qxx_0\right\}} \tag{30}$$

It will be useful to notice that, by integration by parts, one has

$$\int Dz\, f(z)\,z = \int Dz\, f'(z)\,. \tag{31}$$

This is a bit of a longer computation, try to at least go through the solution and verify that all passages make sense to you.

We need to take a saddle point on q and then do integration by parts. For convenience, define

$$F = -\frac{\lambda q}{2}x^2 + \sqrt{\lambda q}xz + \lambda q x x_0 \tag{32}$$

with the identities

$$\frac{\partial F}{\partial q} = -\frac{\lambda}{2}x^2 + \frac{1}{2}\sqrt{\frac{\lambda}{q}}xz + \lambda x x_0 \tag{33}$$

and

$$\frac{\partial F}{\partial z} = \sqrt{\lambda q}x \tag{34}$$

which allows us to write

$$\mathrm{extr}_q \left[-\frac{\lambda}{4}q^2 + \int DzP_X\left(x_0\right)\mathrm{d}x_0 \log\left(\int P_X\left(x\right)\mathrm{d}x \ \exp\left\{ -\frac{\lambda q}{2}x^2 + \sqrt{\lambda q}xz + \lambda q x x_0 \right\} \right) \right] =$$
$$\mathrm{extr}_q \left[-\frac{\lambda}{4}q^2 + \int DzP_X\left(x_0\right)\mathrm{d}x_0 \log\left(\int P_X\left(x\right)e^F\,\mathrm{d}x \right) \right]$$

In doing the saddle point we need to compute two derivatives. The first one is immediate

$$\frac{\partial}{\partial q}\left(-\frac{\lambda}{4}q^2 \right) = -\frac{\lambda}{2}q \tag{35}$$

The other is quite more involved

$$\begin{aligned}
&\frac{\partial}{\partial q}\int DzP_X\left(x_0\right)\mathrm{d}x_0 \log\left(\int P_X\left(x\right)e^F\,\mathrm{d}x \right) \\
&= \int DzP_X\left(x_0\right)\mathrm{d}x_0 \frac{\int P_X\left(x\right)e^F \frac{\partial F}{\partial q}\,\mathrm{d}x}{\int P_X\left(x\right)e^F\,\mathrm{d}x} \\
&= \int DzP_X\left(x_0\right)\mathrm{d}x_0 \frac{\int P_X\left(x\right)e^F \left(-\frac{\lambda}{2}x^2 + \frac{1}{2}\sqrt{\frac{\lambda}{q}}xz + \lambda x x_0 \right)\mathrm{d}x}{\int P_X\left(x\right)e^F\,\mathrm{d}x}
\end{aligned}$$

We focus just on the piece in the numerator that contains z and massage it using integration by part on z. We have

$$
\begin{aligned}
&\int DzP_X(x_0)\,\mathrm{d}x_0 \frac{\int P_X(x)\,e^F x\,\mathrm{d}x}{\int P_X(x)\,e^F\,\mathrm{d}x} z \\
&= \int DzP_X(x_0)\,\mathrm{d}x_0 \frac{\partial}{\partial z}\frac{\int P_X(x)\,e^F x\,\mathrm{d}x}{\int P_X(x)\,e^F\,\mathrm{d}x} \\
&= \int DzP_X(x_0)\,\mathrm{d}x_0 \frac{\left(\int P_X(x)\,e^F x \frac{\partial F}{\partial z}\,\mathrm{d}x\right)\left(\int P_X(x)\,e^F\,\mathrm{d}x\right) - \left(\int P_X(x)\,e^F x\,\mathrm{d}x\right)\left(\int P_X(x)\,e^F \frac{\partial F}{\partial z}\,\mathrm{d}x\right)}{\left(\int P_X(x)\,e^F\,\mathrm{d}x\right)^2} \\
&= \int DzP_X(x_0)\,\mathrm{d}x_0 \frac{\left(\int P_X(x)\,e^F \sqrt{\lambda q}x^2\,\mathrm{d}x\right)\left(\int P_X(x)\,e^F\,\mathrm{d}x\right) - \left(\int P_X(x)\,e^F x\,\mathrm{d}x\right)\left(\int P_X(x)\,e^F \sqrt{\lambda q}x\,\mathrm{d}x\right)}{\left(\int P_X(x)\,e^F\,\mathrm{d}x\right)^2} \\
&= \sqrt{\lambda q}\int DzP_X(x_0)\,\mathrm{d}x_0 \frac{\left(\int P_X(x)\,e^F x^2\,\mathrm{d}x\right)\left(\int P_X(x)\,e^F\,\mathrm{d}x\right) - \left(\int P_X(x)\,e^F x\,\mathrm{d}x\right)\left(\int P_X(x)\,e^F x\,\mathrm{d}x\right)}{\left(\int P_X(x)\,e^F\,\mathrm{d}x\right)^2} \\
&= \sqrt{\lambda q}\int DzP_X(x_0)\,\mathrm{d}x_0 \left[\frac{\int P_X(x)\,e^F x^2\,\mathrm{d}x}{\int P_X(x)\,e^F\,\mathrm{d}x} - \left(\frac{\int P_X(x)\,e^F x\,\mathrm{d}x}{\int P_X(x)\,e^F\,\mathrm{d}x}\right)^2\right]
\end{aligned}
$$

Let's look at the last piece here. We can use Nishimori:

$$
\int P_X(x_0)\,\mathrm{d}x_0 \left(\frac{\int P_X(x)\,e^F x\,\mathrm{d}x}{\int P_X(x)\,e^F\,\mathrm{d}x}\right)^2 = \int P_X(x_0)\,\mathrm{d}x_0 \frac{\left(\int P_X(x)\,e^F x\,\mathrm{d}x\right)\left(\int P_X(x)\,e^F x_0\,\mathrm{d}x\right)}{\left(\int P_X(x)\,e^F\,\mathrm{d}x\right)^2} \tag{36}
$$

So we have shown that

$$
\int DzP_X(x_0)\,\mathrm{d}x_0 \frac{\int P_X(x)\,e^F xz\,\mathrm{d}x}{\int P_X(x)\,e^F\,\mathrm{d}x} = \sqrt{\lambda q}\int DzP_X(x_0)\,\mathrm{d}x_0 \frac{\int P_X(x)\,e^F (x^2 - xx_0)\,\mathrm{d}x}{\int P_X(x)\,e^F\,\mathrm{d}x} \tag{37}
$$

Which also means that

$$
\begin{aligned}
&\frac{\partial}{\partial q}\int DzP_X(x_0)\,\mathrm{d}x_0 \log\left(\int P_X(x)\,e^F\,\mathrm{d}x\right) \\
&\qquad = \int DzP_X(x_0)\,\mathrm{d}x_0 \frac{\int P_X(x)\,e^F\left(-\frac{\lambda}{2}x^2 + \frac{\lambda}{2}(x^2 - xx_0) + \lambda xx_0\right)\mathrm{d}x}{\int P_X(x)\,e^F\,\mathrm{d}x} \\
&\qquad = \frac{\lambda}{2}\int DzP_X(x_0)\,\mathrm{d}x_0 \frac{\int P_X(x)\,e^F xx_0\,\mathrm{d}x}{\int P_X(x)\,e^F\,\mathrm{d}x}
\end{aligned}
$$