

Statistical Physics of Computation 2025 - Exercises

Vittorio Erba, Emanuele Troiani

Week 5

5.1 Replica computation for the spiked-Wigner model

We are given as observation a $N \times N$ symmetric matrix Y created as

$$Y = \sqrt{\frac{\lambda}{N}} \underbrace{x^* x^{*\top}}_{N \times N \text{ rank-one matrix}} + \underbrace{\xi}_{\text{symmetric iid noise}}$$

where $x^* \in \mathbb{R}^N$ with $x_i^* \stackrel{\text{i.i.d.}}{\sim} P_X(x)$, $\xi_{ij} = \xi_{ji} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$ for $i \leq j$. We keep the prior P_X generic, as long as it is factorized over all the components of the vector x .

Our task shall be to recover the vector x from the knowledge of Y , the signal-to-noise ratio λ and the prior P_X . As we saw during the lecture, this can be achieved using the posterior estimation, i.e. by computing the posterior distribution and evaluating some statistics over it.

In this exercise, you will compute the normalization factor of the posterior distribution, i.e. the partition function of the problem, and derive the state equation for the overlap order parameter.

1. Show that the posterior distribution $P(x|Y)$ for the problem can be written as

$$P(x|\mathbf{Y}) = \frac{1}{Z(\mathbf{Y})} \left[\prod_{i=1}^N P_X(x_i) \right] \left[\prod_{i \leq j} \frac{e^{-\frac{\lambda}{2N} x_i^2 x_j^2 + \sqrt{\frac{\lambda}{N}} x_i x_j y_{ij}}}{\sqrt{2\pi}} \right] \quad (1)$$

for a specific $Z(\mathbf{Y})$. How is $Z(\mathbf{Y})$ defined for this measure?

We are interested in computing the averaged free entropy associated to the posterior distribution, i.e.

$$\lim_{N \rightarrow \infty} \mathbb{E}_Y \left[\frac{1}{N} \log Z(Y) \right]$$

which we can compute using the replica method by computing first

$$\mathbb{E}_Y Z(Y)^n. \quad (2)$$

2. Show that the averaged replicated partition function equals

$$\mathbb{E}_Y [Z^n] = \int dY e^{-\frac{1}{2} \sum_{i \leq j} y_{ij}^2} \prod_{\alpha=0}^n \int dx^{(\alpha)} \left(\prod_{i=1}^N P_X(x_i^{(\alpha)}) \right) \left(\prod_{i \leq j} \frac{e^{-\frac{\lambda}{2N} (x_i^{(\alpha)})^2 (x_j^{(\alpha)})^2 + \sqrt{\frac{\lambda}{N}} x_i^{(\alpha)} x_j^{(\alpha)} y_{ij}}}{\sqrt{2\pi}} \right) \quad (3)$$

where you should notice that we are taking the product $n + 1$ replicas.

3. Integrate over the disorder, i.e. the observation Y , to get at leading order in N

$$\mathbb{E}_Y[Z^n] = \int \prod_{\alpha,i} P_X(x_i^{(\alpha)}) dx_i^{(\alpha)} \exp\left(\frac{\lambda N}{2} \sum_{\alpha < \beta} \left(\sum_i \frac{x_i^{(\alpha)} x_i^{(\beta)}}{N}\right)^2\right) \quad (4)$$

4. Introduce the order parameter

$$q_{\alpha\beta} = \frac{1}{N} \sum_i x_i^\alpha x_i^\beta \quad (5)$$

using a Dirac delta function, and its conjugate \hat{q} through the Fourier representation of the Dirac delta, and obtain

$$\mathbb{E}_Y[Z^n] = \int \prod_{\alpha < \beta} d\hat{q}_{\alpha\beta} dq_{\alpha\beta} \exp(N I_{\text{energy}}(q_{\alpha\beta}, \hat{q}_{\alpha\beta}) + N I_{\text{entropy}}(\hat{q}_{\alpha\beta})) \quad (6)$$

where we defined

$$I_{\text{energy}}(q_{\alpha\beta}, \hat{q}_{\alpha\beta}) = \frac{\lambda}{2} \sum_{\alpha < \beta} q_{\alpha\beta}^2 - \sum_{\alpha < \beta} q_{\alpha\beta} \hat{q}_{\alpha\beta} \quad (7)$$

and

$$I_{\text{entropy}}(\hat{q}_{\alpha\beta}) = \log\left(\int \prod_{\alpha} P_X(x_{\alpha}) dx_{\alpha} \exp\left\{\sum_{\alpha < \beta} \hat{q}_{\alpha\beta} x_{\alpha} x_{\beta}\right\}\right) \quad (8)$$

where we stress that here the integral over dx_{α} runs over the real numbers for all $\alpha = 0, \dots, n$.

5. Show that in the RS ansatz $q_{\alpha\beta} = q$, $\hat{q}_{\alpha\beta} = \hat{q}$, the energetic term satisfies

$$\lim_{n \rightarrow 0} \frac{1}{n} I_{\text{energy}}(q_{\alpha\beta}, \hat{q}_{\alpha\beta}) = \frac{\lambda}{4} q^2 - \frac{q\hat{q}}{2} \quad (9)$$

for q and \hat{q} real numbers.

6. We now need to compute the entropic term I_{entropy} in the RS ansatz. This is more tricky. We start by proving an identity that will come useful later. Show that for any positive constant a we have the following identity:

$$\exp\left\{\frac{ax^2}{2}\right\} \propto \int \exp\left\{-\frac{z^2}{2a} + zx\right\} dz \quad (10)$$

Typically, we call this the Hubbard-Stratonovich (HS) transformation.

7. The HS transformation is particularly useful in spin glasses, as we can use it to "decouple" strongly interacting systems at the price of introducing a random external field z . We can see it in practice in this example. Show that

$$\exp\left\{\sum_{i=1}^N \frac{x_i^2}{2} + \sum_{i,j} x_i x_j\right\} \propto \prod_{i=1}^N \int \exp\left\{-\frac{z^2}{2} + zx_i\right\} dz \quad (11)$$

Here by decoupling we mean that the left hand side is not a product over i independent terms (not factorized), as for all pairs i, j the variables x_i and x_j interact (if we imagine the exponent to be an energy function). The right hand side instead is factorized over i , modulo paying the price of a Gaussian integral. **(Bonus)** Try to use the HS transform to re-derive the Curie-Weiss model partition function without using any Dirac delta function.

8. Show that in the RS ansatz $q_{\alpha\beta} = q$, $\hat{q}_{\alpha\beta} = \hat{q}$, the entropic term satisfies

$$I_{\text{entropy}}(\hat{q}_{\alpha\beta}) = \log \left(\int Dz \left[\left(\int P_X(x) dx \exp \left\{ -\frac{\hat{q}}{2} x^2 + \sqrt{\hat{q}} x z \right\} \right)^{n+1} \right] \right) \quad (12)$$

for \hat{q} real numbers. In the derivation, you will have to use the RS ansatz followed by the HS transform.

9. Show that in the small n limit, the entropic term satisfies

$$\lim_{n \rightarrow 0} \frac{1}{n} I_{\text{entropy}}(\hat{q}_{\alpha\beta}) = \int Dz I(\hat{q}, z) \log(I(\hat{q}, z)) \quad (13)$$

where we defined $I(\hat{q}, z)$ as

$$I(\hat{q}, z) = \int P_X(x) dx \exp \left\{ -\frac{\hat{q}}{2} x^2 + \sqrt{\hat{q}} x z \right\} \quad (14)$$

10. Show that the previous expression can be rewritten as

$$\lim_{n \rightarrow 0} \frac{1}{n} I_{\text{entropy}}(\hat{q}_{\alpha\beta}) = \int Dz \int P_X(x_0) dx_0 \log \left(\int P_X(x) dx \exp \left\{ -\frac{\hat{q}}{2} x^2 + \sqrt{\hat{q}} x z + \hat{q} x x_0 \right\} \right) \quad (15)$$

11. Argue finally that the free entropy

$$\phi = \lim_{N \rightarrow \infty} \mathbb{E}_Y \left[\frac{1}{N} \log Z(Y) \right] \quad (16)$$

can be expressed as

$$\phi = \text{extr}_{q, \hat{q}} \left[\frac{\lambda}{4} q^2 - \frac{q \hat{q}}{2} + \int Dz \int P_X(x_0) dx_0 \log \left(\int P_X(x) dx \exp \left\{ -\frac{\hat{q}}{2} x^2 + \sqrt{\hat{q}} x z + \hat{q} x x_0 \right\} \right) \right] \quad (17)$$

or equivalently as

$$\phi = \text{extr}_q \left[-\frac{\lambda}{4} q^2 + \int Dz P_X(x_0) dx_0 \log \left(\int P_X(x) dx \exp \left\{ -\frac{\lambda q}{2} x^2 + \sqrt{\lambda q} x z + \lambda q x x_0 \right\} \right) \right] \quad (18)$$

12. (**Bonus**) Show that the state equations satisfies

$$q = \int Dz P_X(x_0) \frac{\int P_X(x) x x_0 dx \exp \left\{ -\frac{\lambda q}{2} x^2 + \sqrt{\lambda q} x z + \lambda q x x_0 \right\}}{\int P_X(x) dx \exp \left\{ -\frac{\lambda q}{2} x^2 + \sqrt{\lambda q} x z + \lambda q x x_0 \right\}} \quad (19)$$

It will be useful to notice that, by integration by parts, one has

$$\int Dz f(z) z = \int Dz f'(z). \quad (20)$$

This is a bit of a longer computation, try to at least go through the solution and verify that all passages make sense to you.