Statistical Physics of Computation 2025 - Exercises

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Week 11

11.1 Matching

Consider the matching problem on graphs. Use the factor graph model representation from the previous homework.

1. Write belief propagation equations for the probability distribution

$$P\left(\{S_{(ij)}\}_{(ij)\in E}\right) = \frac{1}{Z(\beta)} \prod_{(ij)\in E} e^{\beta S_{(ij)}} \prod_{i=1}^{N} \mathbb{I}\left(\sum_{j\in\partial i} S_{(ij)} \leq 1\right) \tag{1}$$

   where $S_{(ij)} \in \{0,1\}$ denotes whether a link (ij) is selected for the given matching or not, and E is the edge set of the given graph. Be careful that in the matching problem the nodes of the graph play the role of factor nodes in the factor graph model and edges in the graph carry the variable nodes in the factor graph model.

2. Write the corresponding Bethe free entropy in order to estimate $\log Z(\beta)$.

3. Describe how to estimate the number of matchings of a given size on a given randomly generated large graph G.

4. Consider now that the graph is a uniformly distributed d-regular random graph. Write the Replica Symmetric BP equations (i.e. the BP equations under the uniform neighborhood ansatz), and obtain the single scalar equation

$$\frac{1-a}{a} = 1 + (d-1)\frac{a}{1-a}e^{\beta} \tag{2}$$

   where $a = \psi_1$.

5. Solve the equation found in the previous point, and compute the Bethe free entropy and the matching size (energy) as functions of β.

6. Plot the entropy as a function of the energy for different degrees d, and comment the curves.

7. Consider now the measure

$$P\left(\{S_{(ij)}\}_{(ij)\in E}\right) = \frac{1}{Z} \prod_{i=1}^{N} \mathbb{I}\left(\sum_{j\in\partial i} S_{(ij)} \leq 1\right) \tag{3}$$

   i.e. the $\beta = 0$ case of the previous measure, still for random d-regular graphs. What is the size of a matching extracted randomly from this measure at leading order in the $N \to \infty$ limit, and how many matchings of that size are there at leading order?