Statistical Physics of Computation 2025 - Exercises

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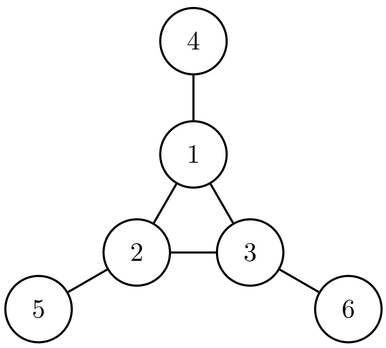
Week 10

10.1 Factor graph representations

1. **Independent set problem**

   The independent set problem is a problem defined and studied in combinatorics and graph theory. Given a (unweighted, undirected) graph $G(V, E)$, an independent set $S \subseteq V$ is defined as a subset of nodes such that if $i \in S$ then for all $j \in \partial i$ we have $j \notin S$. In other words in for all $(ij) \in E$ only i or j can belong to the independent set.

   Example graph:

   (a) Find a bijection between the set of subsets $S \subseteq V$ with $|V| = N$ (where $|V|$ denotes the cardinality of the node set V) and $\{0, 1\}^N$.

   (b) Write a probability distribution that is uniform over all independent sets on a given graph (represented as elements of $\{0, 1\}^{|V|}$, and represent it as a factor graph in the example given above.

   (c) Write a probability distribution that gives a larger weight to larger independent sets, where the size of an independent set is simply its cardinality $|S|$. Represent it as a factor graph for the example given above.
   Hint: many probability distributions assign more weight to $|S|$, but some choices lead to simpler factor graphs...

   (d) Write the Belief Propagation equations for these problems (without coding or solving them) and the expression for the Bethe free energy that would be computed from the BP fixed points.

2. **Matching problem**

The matching problem is another classical problem of graph theory. It is related to the dimer problem in statistical physics, where you aim at covering a graph with two-site dimers. Given a (unweighted, undirected) graph $G(V, E)$ a matching $M \subseteq E$ is defined as a subset of edges such that if $(ij) \in M$ then no other edge that contains node i or j can be in M. In other words a matching is a subset of edges such that no two edges of the set share a node.

Example problem: same as the independent graph one.

(a) Write a probability distribution that is uniform over all matchings on a given graph, and draw the factor graph corresponding to the example graph given for the independent set problem. Hint: again, find a Boolean encoding for a matching, similarly as what we did for the independent set.

(b) Write a probability distribution that that gives a larger weight to larger matchings, where the size of a matching is the cardinality $|M|$. Then, draw the factor graph corresponding to the example graph given for the independent set problem

(c) Write the Belief Propagation equations for these problems (without coding or solving them) and the expression for the Bethe free energy that would be computed from the BP fixed points.

10.2 The Ising model on d-regular random graphs

Consider the following probability distribution

$$p_G(S) = \frac{1}{Z} \exp\left(\beta \sum_{(ij)\in E} S_i S_j \right) \tag{1}$$

where G is a graph with N nodes, and edge set E. This is the Ising model on a graph G. Assume that G is a uniformly sampled d-regular graph, i.e. a graph whose nodes have all $d = O(1)$ neighbours.

1. Sketch the associated factor graph, and write the BP equations.

2. When studying problems on random d-regular graphs, we often make a sort of RS ansatz by saying that the BP messages will be uniform over all edges of the graph. This is because all local neighbourhoods on the factor graphs are identical. Similarly as to an RS ansatz, this is not necessarily the only, or the correct solution to the BP equations, but for sure it is the simplest to study and worth looking at. Use this uniformity assumption to derive the following reduced BP equation

$$\chi(s) = \frac{\left[\sum_t e^{\beta s t}\chi(t)\right]^{d-1}}{\sum_{s'}\left[\sum_t e^{\beta s' t}\chi(t)\right]^{d-1}} \tag{2}$$

   where for all nodes i and edges (ij) we called $\chi = \chi^{i\to(ij)}$.

3. Compute the marginal over a single spin s under the uniform ansatz.

4. Show that the paramagnetic fixed point $\chi(s) = 1/2$ is a solution of the BP equations for all β. Why do we call this paramagnetic?

5. Show that the ferromagnetic fixed point $\chi(+1) = a \in [0,1]$ and $\chi(-1) = 1-a$ is a solution of the BP equations for some value of a_*, and show that a_* satisfies a scalar equation.

6. Compute the average magnetization as a function of a^*.

7. We expect that in this model there is a second order phase transition, as this is the case for $d \to \infty$ (Curie-Weiss model), as well as for the finite dimensional counterparts of the Ising model. To derive the second order phase transition threshold β_c we can study the stability of the iteration

$$a_{t+1} = \left[1 + \left(\frac{a_t e^{-\beta} + (1-a_t)e^{\beta}}{a_t e^{\beta} + (1-a_t)e^{-\beta}} \right)^{d-1} \right]^{-1} = f(a_t) \tag{3}$$

   around the paramagnetic solution $a = 1/2$. We expect that the iteration will fall back on the paramagnetic solution in the paramagnetic phase, while it will diverge away from it in the ferromagnetic phase. In other words, $a = f(a)$ is our state equation, and we are checking whether $a^* = 1/2$ is a maximum of the associated free entropy.
   Argue that the iteration initialized at $a_0 = 1/2 + \epsilon$ for small ϵ converges back to the paramagnetic solution only if $f'(1/2) < 1$.

8. Compute the critical threshold $\beta_c(d)$ as a function of the degree d.

9. Compare the value of $\beta_c(d)$ with the value for the phase transition of the 1d Ising model $\beta_c = +\infty$, and with the value for the Curie-Weiss model $\tilde{\beta}_c = 1$ (notice that there is a difference in normalization between this Ising model and the Curie-Weiss model!). To which values of d the two correspond? How should we rescale β in our problem to be in the same scaling as the Curie-Weiss model?