Gauge Theories and the Standard Model

Problem Set 7

Due Tuesday, November 4, in class (BSP 727)

Lecture: Marc Riembau
Exercises: Andrea Luzio, Barak Gabai

Problem 1: Muon decay in the Electroweak theory

(i) Consider the (dominant) three-body decay of a muon to an electron and neutrinos

$$\mu^-(P) \to e^-(p)\bar{\nu}_e(q_1)\nu_\mu(q_2)\,.$$

Draw the Feynman diagram(s) that contribute to this process in the SM. Neglect the electron mass with respect to the muon mass and compute the amplitude square by averaging over initial and summing of final fermion polarizations.

Since $m_\mu \simeq 106\,\mathrm{MeV}$, which is much smaller than $m_W \simeq 80\mathrm{GeV}$, you should neglect the muon mass compared with the W boson mass. This corresponds to replacing the W propagator by

$$\langle 0|W^+(p)W^-(-p)|0\rangle = \frac{-i}{p^2 - m_W^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m_W^2}\right) \;\to\; \frac{i}{m_W^2}\eta_{\mu\nu}.$$

Work in this limit.

(ii) In the rest frame of the muon the differential width is then obtained by the matrix-element square times the three-body phase-space element

$$d\Gamma = \frac{(2\pi)^4}{2m_\mu}|\mathcal{M}|^2\delta^4(P-p-q_1-q_2)\frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_e}\frac{d^3\boldsymbol{q}_1}{(2\pi)^3 2E_1}\frac{d^3\boldsymbol{q}_2}{(2\pi)^3 2E_2}\,.$$

Show that for the muon decay the result is

$$d\Gamma = \frac{4G_\mathrm{F}^2}{(2\pi)^5 m_\mu}(q_2\cdot p)(q_1\cdot P)\delta^4(P-p-q_1-q_2)\frac{d^3\boldsymbol{p}}{E_e}\frac{d^3\boldsymbol{q}_1}{E_1}\frac{d^3\boldsymbol{q}_2}{E_2}\,.$$

Work out what G_F is in terms of the $SU(2)$ gauge coupling g and the W boson mass m_W, and also in terms of the Higgs vev v.

(iii) Integrate first over the phase space of the neutrino momenta. To this end define and evaluate the corresponding tensor integral

$$I^{\mu\nu} = \int \frac{d^3\boldsymbol{q}_1}{E_1}\frac{d^3\boldsymbol{q}_2}{E_2}q_1^\mu q_2^\nu\delta(P-p-q_1-q_2)\,.$$

What are the transformation properties of $I^{\mu\nu}$ under Lorentz? On which linear combination of momenta can the integral only depend on? To evaluate $I^{\mu\nu}$ decompose it in terms of independent Lorentz structures.

(Hint: Remember that Lorentz scalars are independent of the reference frame, so you may evaluate them without loss of generality in the centre-of-mass frame. You should find that

$$I^{\mu\nu} = \frac{\pi}{6}(\eta^{\mu\nu}q^2 + 2q^\mu q^\nu)$$

with q the only momentum on which $I^{\mu\nu}$ can depend on.)

(iv) Integrate also over the allowed phase space of the electron and show that the width is

$$\Gamma_\mu = \frac{G_\mathrm{F}^2 m_\mu^5}{192\pi^3}$$

(v) Use the experimental input for muon lifetime from `pdglive.lbl.gov` to determine G_F. Compare your value of G_F to the value that PDG lists and discuss which are the approximations that could account for the difference in the values.