

Gauge Theories and the Standard Model

Problem Set 5

Due Tuesday, October 14, in class (BSP 727)

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Problem 1: Spontaneous breaking in the linear $SU(2) \times SU(2)$ σ -model

Consider the theory of four real scalar fields, Φ_1, Φ_2, Φ_3 and Φ_4 , that is invariant under a global $SO(4)$ symmetry. Using the isomorphism $SO(4) \cong SU(2)_L \times SU(2)_R$ we can rewrite the real scalars in terms of the 2×2 matrix

$$\Sigma = \frac{1}{\sqrt{2}}(\mathbb{I}_2\Phi_4 + i\sigma_a\Phi_a).$$

Σ transforms as a $(\mathbf{2}, \mathbf{2})$ under the global symmetry $SU(2)_L \times SU(2)_R$ via

$$\Sigma \longrightarrow U_L \Sigma U_R^\dagger,$$

with $U_L \in SU(2)_L$ and $U_R \in SU(2)_R$.

- (i) Show that Σ is “pseudo-real”, namely that

$$\Sigma^* = \sigma_2 \Sigma \sigma_2$$

and verify that this property is respected by the transformation above.

- (ii) Show that at the dimension-four level the most general Lagrangian that is invariant under $SU(2)_L \times SU(2)_R$ is

$$\mathcal{L} = \frac{1}{2}\text{Tr}[\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] + \frac{1}{2}\mu^2 \text{Tr}[\Sigma \Sigma^\dagger] - \frac{1}{4}\lambda \text{Tr}[\Sigma \Sigma^\dagger \Sigma \Sigma^\dagger].$$

What about invariants involving the determinant and terms with epsilon-tensor contractions?

(Hint: Work with the components $\Sigma_{i_L}^{i_R}$ and remember that $SU(2)$ has two invariant tensors.)

- (iii) If μ^2 is positive the theory exhibits spontaneous-symmetry breaking. Pick a vacuum of your liking and expand the Lagrangian around it. How many massless states (Goldstone bosons) do you find in the theory?

(Hint: It is easier to choose the vacuum if you express Σ in terms of Φ_i s.)

- (iv) For the vacuum that you picked, find $\langle \Sigma \rangle$ and find the broken and unbroken generators of $SU(2)_L \times SU(2)_R$ by proceeding similarly to the proof of the Goldstone Theorem. You can of course choose the vacuum to align with Φ_4 , but you might want to work out the case where the vacuum that you choose is not pointing along the Φ_4 direction.

- (v) What is the symmetry breaking pattern that you found, which symmetry is the vacuum still preserving?