

# Gauge Theories and the Standard Model

## Problem Set 3

Due Tuesday, September 30, in class (BSP 727)

**Lecture:** Marc Riembau

**Exercises:** Andrea Luzio

### Problem 1: Massive vector propagators from the path integral

Consider the theory of a massive vector,  $A_\mu$ , with mass  $m$  coupled to a scalar,  $\phi$ , via a kinetic mixing interaction. In generic  $\xi$  gauge the Lagrangian in Minkowski space reads:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial^\mu\phi - mA^\mu)(\partial_\mu\phi - mA_\mu) + \mathcal{L}_{\text{gauge-fix}},$$
$$\mathcal{L}_{\text{gauge-fix}} = -\frac{1}{2\xi}(\partial_\mu A^\mu + \tilde{m}\xi\phi)^2.$$

In this problem we will derive the Feynman propagator,  $\Delta_F(x-y)$ , for this theory using the path integral formalism.

(i)

Compute the Euclidean action,  $S_E$ , of this Lagrangian. Remember that to transform to Euclidean space both coordinates and vector field need to be transformed via

$$\begin{array}{ll} x^0 \rightarrow -ix_E^0, & x_0 \rightarrow -ix_0^E, \\ x^i \rightarrow x_E^i, & x_i \rightarrow -x_i^E, \\ \partial^0 \rightarrow i\partial_E^0, & \partial_0 \rightarrow i\partial_0^E, \\ \partial^i \rightarrow -\partial_E^i, & \partial_i \rightarrow \partial_i^E, \\ A^0 \rightarrow iA_E^0, & A_0 \rightarrow iA_0^E, \\ A^i \rightarrow -A_E^i, & A_i \rightarrow A_i^E. \end{array}$$

Below, we do not keep track of all the labels “ $E$ ” that indicate that the quantities are Euclidean.

(ii)

Package all real fields into one field

$$\Phi(x) = (A_0^E(x), A_1^E(x), A_2^E(x), A_3^E(x), \phi(x))^T$$

and compute the kinetic matrix  $K_{\{M,x\};\{N,y\}}$ , defined via

$$S_E = \frac{1}{2} \int d^4x d^4y \Phi_M(x) K_{\{M,x\};\{N,y\}} \Phi_N(y).$$

(iii)

Compute the matrix  $\tilde{K}(p)$  that is needed to obtain the Feynman propagator in Fourier space as

$$\Delta_{MN}^E(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} (\tilde{K}^{-1}(p))_{MN}.$$

(iv)

For the specific case of  $\tilde{m} = m$  invert  $\tilde{K}(p)$  to obtain the Euclidean propagator for the vector and scalar field. Continue your result back to Minkowski space and verify by choosing  $\xi$  and  $\tilde{m}$  appropriately that you recover

- the photon propagator in generic  $\xi$  gauge, and
- the propagator of a massive vector in unitary gauge.

(v)

For generic  $\tilde{m}$  the propagator does not split into a block-diagonal form. It is instead a  $5 \times 5$  matrix in the space of fields. In this case, inverting the matrix  $\tilde{K}(p)$  is trickier. One way of doing this is to decompose  $\tilde{K}(p)$  into transverse and longitudinal parts. Namely

$$\tilde{K}(p) = AP_T + P_i X^i_j P^{j\dagger},$$

with

$$P_T = \begin{pmatrix} g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} & 0 \\ 0 & 0 \end{pmatrix} \quad P_1 = \begin{bmatrix} i \frac{p_\mu}{|p|} \\ 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Compute  $A$  and the  $2 \times 2$  matrix  $X$ . Do you see why this decomposition is useful? **Hint:** show that

$$\tilde{K}^{-1}(p) = A^{-1}P_T + P_i (X^{-1})^i_j P^{j\dagger}.$$

(vi)

Compute  $\tilde{K}^{-1}(p)$  and find the Feynman propagator in Fourier space. Continue the result to Minkowski space and verify that you reproduce your previous computation when  $\tilde{m} = m$ .

**Hint:** after the rotation to Minkowski space you should find

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \hat{K}^{-1}(p),$$

with

$$\hat{K}^{-1}(p) = \begin{bmatrix} \frac{-i}{p^2 - m^2} \left( g_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2 - \xi m \tilde{m}} \left( 1 + \frac{\xi(m - \tilde{m})}{p^2 - \xi m \tilde{m}} \left( \frac{m - \tilde{m}}{1 - \xi} + \tilde{m} \right) \right) \right) & -\xi \frac{(m - \tilde{m})p_\mu}{(p^2 - \xi m \tilde{m})^2} \\ \xi \frac{(m - \tilde{m})p_\nu}{(p^2 - \xi m \tilde{m})^2} & i \frac{p^2 - \xi m^2}{(p^2 - \xi m \tilde{m})^2} \end{bmatrix}.$$

You probably now realise why most people enjoy working in Feynman gauge, i.e.,  $\tilde{m} = m$  and  $\xi = 1$ .