Gauge Theories and the Standard Model

Problem Set 2

Due Tuesday, September 23, in class (BSP 727)

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Problem 1: QED, forward/scalar photons and gauge invariance

In the lecture you discussed the Ward identity for QED, which ensures that

$$k_\mu \, \mathcal{M}^\mu = 0 \, ,$$

where $\mathcal{M}^\mu$ denotes a generic Feynman amplitude with at least one photon external leg, with momentum k. The identity implies that the unphysical "forward" photon (also known as "scalar" photon), namely the one with polarization vector proportional to k, has vanishing amplitudes. In this exercise we will show that forward (or scalar) photons are not produced in a particular reaction.

Address the following points:

(i) Check that the amplitude for producing a forward photon γ_F out of e^+e^-, i.e. the process

$$e^+e^- \to \gamma_T \, \gamma_F \, ,$$

vanishes exactly. In the above equation, γ_T denotes an ordinary physical photon, with plus or minus helicity. Notice that there is no need to employ the explicit expressions for the spinor wave functions and for the physical photon polarisation vector. The result can be immediately obtained from the equations of motions of the spinor wave functions.

(ii) Introduce in the QED theory a new state, with the same charge as the electron: the muon, μ, with mass $m_\mu = 106$ MeV. Write down the most general Lagrangian, with terms up to energy dimension 4, that is compatible with the $U(1)$ gauge symmetry. Show that the term

$$\mathcal{L}_{\mu e}(x) = \widetilde{e} \, \bar{\Psi}^{(\mu)}(x) \, A_\mu(x) \, \gamma^\mu \, \Psi^{(e)}(x) + \text{h.c.} \, ,$$

where "h.c." denotes the Hermitian conjugation and $\widetilde{e}$ is a new hypothetical free parameter of the theory, breaks the gauge symmetry. Therefore, it cannot appear in the gauge invariant Lagrangian.

(iii) Ignore for now that $\mathcal{L}_{\mu e}$ breaks gauge invariance and thus it leads to an inconsistent theory, and add it to the Lagrangian. Show that the amplitude of

$$e^+e^- \to \gamma_T \, \gamma_F$$

is now different from zero.

(iv) You should have now discovered that if $m_\mu = m_e$ the scalar photon production amplitude vanishes, even if $\mathcal{L}_{\mu e}$ is present. Does this mean that if $m_\mu = m_e$ we could have a consistent theory for the photon without gauge invariance? Or would instead (as it seems more likely) the theory be still in some way gauge invariant even in the presence of the $\mathcal{L}_{\mu e}$ interaction?

(v) The coupling $\mathcal{L}_{\mu e}$ mediates the decay of the muon to an electron and a photon. Neglect the electron mass and compute the decay rate for a muon at rest

$$\mu^- \to e^- \, \gamma_T \,,$$

averaged over the initial muon polarizations and summed over the ones of the final states. Consider only the two transverse (physical) polarizations, $T = \pm 1$ for the final-state photon. The observed life-time of the muon is $\tau = 1/\Gamma_{\text{tot}} \simeq 2.2 \times 10^{-6}$ s, and the experimental bound on its branching fraction to electron and photon

$$\text{BR}(\mu \to e\gamma) \equiv \frac{\Gamma(\mu \to e\gamma)}{\Gamma_{\text{tot}}} < 4.2 \times 10^{-13} \,,$$

at 90% C.L. Use this to extract an experimental bound on the coupling e_e.

Hint: Notice that in the center of mass frame the transverse polarization vectors ϵ_T^μ of the photon (which are just the standard $\lambda = \pm 1$ polarization vectors) are orthogonal to the electron 4-momentum because the electron and the photon 3-momenta are equal and opposite. This simplifies the sum over polarizations. Otherwise you can always perform the complete calculation using the explicit form of the polarization vectors and of the spinor wave functions.

(vi) Instead of $\mathcal{L}_{\mu e}$, consider the $d = 5$ operator

$$\mathcal{L}'_{\mu e} = \frac{e}{\Lambda} \, \bar{\Psi}^{(\mu)} \, \sigma^{\mu\nu} \, \Psi^{(e)} \, F_{\mu\nu} + \text{h.c.} \,.$$

Check that the scalar photon is not produced by this operator. Why? Given the upper limit on $\text{BR}(\mu \to e\gamma)$, compute the lower limit on the operator scale Λ.