

Gauge Theories and the Standard Model

Problem Set 13

Due Tuesday, December 16, in class (BSP 727)

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Problem 1: Neutrino-nucleon scattering

In this problem, we will first (re-)derive the expression we saw in the lecture for deep inelastic electron scattering cross section in terms of the structure functions. We then repeat the similar analysis for neutrino-nucleon scattering.

- (i) Show that the electron nucleon scattering cross section from photon exchange can be written in terms of two Lorentz-invariant structure functions W_2 and W_2 as

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \left(2W_1 \sin^2 \frac{\theta}{2} + W_2 \cos^2 \frac{\theta}{2} \right)$$

where E (E') is the energy of the incoming (outgoing) electron in the nucleon rest frame and θ is the electron scattering angle.

- (ii) Now consider the neutrino-nucleon scattering process, $\nu_l + N \rightarrow l^- + X$. Separate the (charged) weak current as a sum of the leptonic and hadronic parts,

$$J^\lambda = J_l^\lambda + J_h^\lambda.$$

What is the explicit form of the leptonic part, J_l^λ ?

- (iii) Considering the hadronic part, define

$$W_{\alpha\beta}^{(\nu)} = \frac{1}{4M} \Sigma_\sigma \int \frac{d^4x}{2\pi} e^{iq \cdot x} \langle p, \sigma | [J_{h\beta}^\lambda(x), J_{h\alpha}^{\lambda\dagger}(0)] | p, \sigma \rangle.$$

Express $W_{\alpha\beta}^{(\nu)}$ in terms of Lorentz invariant structure functions. How many such functions are there in general?

- (iv) Show that the cross section for neutrino scattering can be written in terms of only three of the Lorentz-invariant structure functions

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{G_F^2}{2\pi^2} E'^2 \left(2W_1^{(\nu)} \sin^2 \frac{\theta}{2} + W_2^{(\nu)} \cos^2 \frac{\theta}{2} - W_3^{(\nu)} \frac{E + E'}{M} \sin^2 \frac{\theta}{2} \right).$$

Problem 2 (Extra): Cartan-Maurer form integral

This is an extra problem to illustrate the calculation of the integral discussed in class. Feel free to skip it entirely.

Consider the integral

$$I[g] = \int_{S^d} dy^1 \dots dy^d \epsilon^{i_1 \dots i_d} \text{tr} \left(g^{-1}(y) \frac{\partial g(y)}{\partial y^{i_1}} \dots g^{-1}(y) \frac{\partial g(y)}{\partial y^{i_d}} \right)$$

where S^d is the sphere and y^1, \dots, y^d some coordinates on it. Here g is an element of a Lie group in some representation, which depends on the point y . In the following you will prove some properties of this integral, as well as the explicit value for a particular case.

- (i) Using the antisymmetry of $\epsilon^{i_1 \dots i_d}$, with $\epsilon^{12 \dots d} = 1$, together with the cyclicity of the trace, show that $I[g]$ vanishes for even d .
- (ii) Show also that the form of the integral is independent of the coordinate system used.
- (iii) Show that the integral is invariant under small deformations of g . In particular, using $g \rightarrow g + \delta g$ show that each factor $g^{-1} \partial_i g$ contributes equally to $\delta I[g]$. Show that, since S^d is compact, $\delta I = 0$.

Hint: Recall that g is a matrix. Use $\delta(g^{-1}g) = 0$ and show that $\delta(g^{-1} \partial_i g) = g^{-1} \partial_i (\delta g g^{-1}) g$.

- (iv) Doing a transformation in the Lie group with parameters y and a transformation with parameters x is equivalent to a transformation with some parameters $y' = y'(y, x)$. In terms of a matrix representation, $g(x)g(y) = g(y'(x, y))$. By taking a derivative with respect y' , show that

$$\frac{y^i}{y'^j} g^{-1}(y) \frac{\partial g(y)}{\partial y^i} = g^{-1}(y') \frac{\partial g(y')}{\partial y'^j}.$$

- (v) Using the result above, write the integrand at a point y' as

$$\det \left(\frac{\partial y}{\partial y'} \right) \epsilon^{i_1 \dots i_d} \text{tr} \left(g^{-1}(y) \frac{\partial g(y)}{\partial y^{i_1}} \dots g^{-1}(y) \frac{\partial g(y)}{\partial y^{i_d}} \right)$$

- (vi) Show that one can write the original integral as

$$I[g] = \epsilon^{i_1 \dots i_d} \text{tr} \left(g^{-1}(y) \frac{\partial g(y)}{\partial y^{i_1}} \dots g^{-1}(y) \frac{\partial g(y)}{\partial y^{i_d}} \right) \frac{1}{\sqrt{\det \gamma(y)}} \int_{S^d} d^d y' \sqrt{\det \gamma(y')}$$

where the matrix γ_{ij} is defined as

$$\gamma_{ij} = -\frac{1}{2} \text{tr} \left(g^{-1}(y) \frac{\partial g(y)}{\partial y^i} \dots g^{-1}(y) \frac{\partial g(y)}{\partial y^j} \right).$$

Hint: Show that $\gamma_{ij}(y') = \frac{\partial y^k}{\partial y'^i} \frac{\partial y^l}{\partial y'^j} \gamma_{kl}(y)$ and take the determinant.

Since the parameter x of the transformation is arbitrary, one can take y and y' to be completely independent. Therefore, one can evaluate the previous expression at $y = 0$. Normalizing the generators and coordinates so that $g(y) \sim 1 + i2T^i y^i$ as $y \rightarrow 0$, one obtains

$$I[g] = (2i)^d \epsilon^{i_1 \dots i_d} \text{tr}(T^{i_1} \dots T^{i_d}) \frac{1}{\sqrt{\det \gamma(0)}} \int d^d y' \sqrt{\det \gamma(y')}.$$

We will evaluate this for the case of $d = 3$ and the Lie group being $SU(2)$. A general element of $SU(2)$ can be written as

$$g(y) = y^4 + 2iy^i T^i$$

with $T^i = \frac{1}{2}\sigma^i$ with σ^i being the Pauli matrices, and $(y^4)^2 = 1 - \vec{y}^2$.

(vii) Show that

$$\gamma_{ij} = \delta_{ij} + \frac{y^i y^j}{1 - \vec{y}^2}$$

so that

$$\det \gamma(y) = \frac{1}{1 - \vec{y}^2}$$

Hint: Decompose $g^{-1} \partial_i g$ in terms of $\alpha \delta_{ij} + \vec{\beta} \cdot \vec{\sigma}$, you should find that $\alpha = 0$.

(viii) Putting everything together, show that

$$I = 24\pi^2$$