

Gauge Theories and the Standard Model

Problem Set 1

Due Tuesday, September 16, in class (BSP 727)

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Problem 1: Dirac algebra and helicity spinors

The Dirac matrices are the matrices that form a matrix representation of the Clifford algebra. Their defining property is

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

We will work in the Weyl / chiral representation in which

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where σ^i are the Pauli matrices and $i = 1, 2, 3$.

In constructing Lorentz invariants it is useful to construct out of these four γ matrices a 5th one that anticommutes with the rest, i.e.

$$\{\gamma^5, \gamma^\mu\} = 0.$$

In the Weyl representation it reads

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

(i) Trace relations

Prove the following trace relations for the γ matrices:

$$\begin{aligned} \text{Tr}[\gamma^\mu] &= 0, \\ \text{Tr}[\gamma^{\mu_1} \dots \gamma^{\mu_n}] &= 0 \quad \text{if } n \text{ is odd,} \\ \text{Tr}[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu}, \\ \text{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}), \\ \text{Tr}[\gamma^5] &= 0, \\ \text{Tr}[\gamma^5\gamma^\mu\gamma^\nu] &= 0, \\ \text{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= -4ie^{\mu\nu\rho\sigma}. \end{aligned}$$

Dirac field operator

A Dirac fermion is described by the Dirac field operator, which in the helicity basis in our conventions is given by

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E}} \sum_{\lambda} \left(a(p, \lambda) u_{\lambda}(p) e^{-ip \cdot x} + b^{\dagger}(p, \lambda) v_{\lambda}(p) e^{ip \cdot x} \right).$$

The label λ is the helicity, i.e., the projection of the state's spin along the direction of momentum, and is either $+\frac{1}{2}$ or $-\frac{1}{2}$. Here $b^{\dagger}(p, \lambda)$ and $a(p, \lambda)$ are creation and annihilation operators, and the coefficient functions $u(p, \lambda)$ and $v(p, \lambda)$ transform in the spinor representation of the Lorentz group.

For a particle moving in the direction

$$\hat{n} = \sin \theta \cos \phi \hat{e}_x + \sin \theta \sin \phi \hat{e}_y + \cos \theta \hat{e}_z$$

with four-momentum p^{μ} , the spinors are

$$u_{\lambda}(p) = \begin{pmatrix} \omega_{-2\lambda} \chi_{2\lambda} \\ \omega_{+2\lambda} \chi_{2\lambda} \end{pmatrix}, \quad v_{\lambda}(p) = \begin{pmatrix} +2\lambda \omega_{+2\lambda} \chi_{-2\lambda} \\ -2\lambda \omega_{-2\lambda} \chi_{-2\lambda} \end{pmatrix},$$

where $\omega_{\pm} \equiv \sqrt{E \pm |\vec{p}|}$ and

$$\chi_{+} = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix}, \quad \chi_{-} = \begin{pmatrix} -e^{-i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}.$$

Tasks

- (ii) Show that $u(p, \lambda)$ and $v(p, \lambda)$ are indeed helicity eigenstates.
- (iii) Demonstrate that $u(p, \lambda)$ and $v(p, \lambda)$ solve the Dirac equation.
- (iv) Show that in the high-energy limit ($E \gg m$), helicity equals chirality in the sense that helicity eigenstates equal chirality eigenstates found by projection operators

$$P_L \equiv \frac{1 - \gamma^5}{2}, \quad P_R \equiv \frac{1 + \gamma^5}{2}.$$

Problem 2: Polarised Bhabha and $e^+e^- \rightarrow \mu^+\mu^-$ scattering

Equipped with the results of Problem 1 compute the QED polarised cross sections for Bhabha scattering

$$e^+(p_1)e^-(p_2) \rightarrow e^-(p_3)e^+(p_4)$$

for center-of-mass energies for which we can safely neglect the electron mass.

Tasks

- (i) Draw the Feynman diagrams and find the corresponding amplitude.
- (ii) Show that the only processes that can be non-zero are

$$(+-, +-), \quad (-+, -+), \quad (+-, -+), \quad (-+, +-), \quad (++, ++), \quad (--, --).$$

In the notation above the ordering is the same as for the momenta (12; 34) and the signs stand for the helicity.

- (iii) Compute the matrix-element squares for those six cases. You can use either the explicit form of $u(p, \lambda)$ and $v(p, \lambda)$ from Problem 1 or use the projectors $P_{L/R}$ to pick up the desired helicities. (Hint: be careful about interfering amplitudes and their relative sign.)
- (iv) Compute the differential cross section for each case and test whether their averaged sum correctly reproduces the unpolarised cross section for Bhabha scattering:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} \left[\frac{u^2}{s^2} + \frac{s^2}{u^2} + \frac{t^2}{s^2} + \frac{s^2}{t^2} \right],$$

with

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \quad t = (p_3 - p_2)^2 = (p_4 - p_1)^2, \quad u = (p_4 - p_2)^2 = (p_3 - p_1)^2,$$

and θ the angle between the incoming electron and the outgoing positron.

- (v) Repeat the computation for the simpler process

$$e^+(p_1)e^-(p_2) \rightarrow \mu^-(p_3)\mu^+(p_4),$$

assuming the muon mass can be neglected.