Introduction to holography - Lecture IX

· Last time: radial quantiz, state-operator corresp, unitarity bounds

- OPE, reducing n-pt. to lower-pt. f, conformal bootstrap

This time: 2d CFTs (very special, many universal prop. det. by $\overset{\delta}{c}$) conf. anomaly coef.

- corr. f. of stress tensor, Schwarzian derivative, Casimir energy on the cylinder, Cardy's formula

- isometries of AdS_{d+1} spacetimes

Weyl invariance & the conformal anomaly

· usually conformally invariant theories are also Weyl invar ($\Leftarrow$)

· Weyl: partition f. $Z[g]$ invariant under $g_{\mu\nu}(x) \to \Omega^2(x)\, g_{\mu\nu}(x)$, $\forall \underbrace{\Omega(x)}_{\text{local}}$

- defining the stress tensor as $T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_E}{\delta g^{\mu\nu}}$ (eucl.).

$$\delta \ln Z = -\frac{1}{2}\int d^dx \sqrt{g}\, \langle T_{\mu\nu}\rangle_g\, \delta g^{\mu\nu} = \int d^dx \sqrt{g}\, \langle T^\mu{}_\mu\rangle_g\, \sigma(x)$$

$\Omega = e^\sigma \simeq 1+\sigma$

- invariance $\Rightarrow$ $T^\mu{}_\mu = 0$ (up to contact terms)

(by contrast, cond. for conformal invar is $T^\mu{}_\mu = \partial_\mu \partial_\nu L^{\mu\nu}$ $d>2$

$= \Box L$ $d=2$)

$\Delta T_{\mu\nu} \sim \partial^\rho \partial^\sigma L_{\mu\rho\nu\sigma}$) can be generated via the curvature couplings

$$S_{imp} \sim \int d^d x \sqrt{g} \left(R L + R_{\mu\nu} L^{\mu\nu} + R_{\mu\nu\rho\sigma} Y^{\mu\nu\rho\sigma} \right).$$

• in 2d, note that $T^\mu{}_\mu = 2T_{z\bar{z}} + 2T_{\bar{z}z} = 0$

$\Rightarrow$ T_{zz} is holomorphic & $T_{\bar{z}\bar{z}}$ is anti-holo

Conformal / Weyl anomalies

• Weyl-invar theories are automatically conformally invar when restricted to flat sp.

• this is guaranteed by the vanishing of the trace of the stress tensor (in flat space)

• however, one may have a situation in which $T^\mu{}_\mu$ is a function of the backgnd. fields (i.e., the metric), times the identity operator $\Rightarrow$ anomalous breaking of Weyl symm. (c/s symm. that is broken by quantum effects)

• since $T^\mu{}_\mu(x)$ is a local, scalar operator of dimension d, so should be the R

central charge

• $d=2$ $\qquad T^\mu{}_\mu = + \frac{c}{24\pi} R[g]$ (eucl. sign, $-$ in Lorentzian)

(& must vanish in flat sp)

& be diff. invar

& consistent w/ abelian nature of Weyl transf)

• $d=4$ $\qquad T^\mu{}_\mu = \frac{a}{64\pi^2} E_4 + \frac{c}{64\pi^2} C_{\lambda\mu\nu\rho} C^{\lambda\mu\nu\rho} + e_1 R^2 + e_2 \Box R.$

...... compt of stress T.).

$d=2$

- in $d=2$, ∀ metric is rel to $\delta_{\mu\nu}$ by a diffeo + conformal transf. Fi…
  we can write
  $$g_{\mu\nu}(x) = e^{2\sigma(x)}\delta_{\mu\nu} \qquad \text{w/} \quad R = -2e^{-2\sigma}\Box_\delta \ldots$$

- since the Weyl anomaly ⇒ dep. of path int. on σ ⇒ completely det. depe…
  of Z on the 2d metric.

$$\frac{\delta}{\delta\sigma} \ln Z[e^{2\sigma}\delta_{\mu\nu}] = \int d^2x\, e^{2\sigma} \underbrace{\langle T^\mu{}_\mu \rangle}_{+\frac{c}{24\pi}R} \delta\sigma = -\frac{c}{12\pi}\int d^2x\, \Box_\delta\sigma\,\delta\sigma$$

$$\Rightarrow \quad Z[e^{2\sigma}\delta_{\mu\nu}] = e^{-\frac{c}{24\pi}\int d^2x\, \sigma(x)\Box_\delta\sigma(x)}\, Z[\delta_{\mu\nu}].$$

✓ negative mod…

how the partition f. dep. on the scale factor

- the prefactor can be rewritten ... in terms ... ...

- another thing one may deduce is how $T_{\mu\nu}$ transforms under conformal transf.

- in the exercise you are asked to show that $\langle T_{zz} \rangle_{e^{2\sigma}\delta_{\mu\nu}} = \frac{c}{12\pi}\left[\partial_z^2\sigma - (\partial_z\sigma)^2\right]$

  do finite directly $e^{2\sigma} = \partial_z \tilde{z}$

- remember in 2d conf. transf. are given by $z \to z + f(z)$ $\bar{z} \to \bar{z} + \bar{f}(\bar{z})$ (infinitesimal)

  under such a transf. $\delta g_{z\bar{z}} = -\partial_z \xi_{\bar{z}} - \partial_{\bar{z}} \xi_z = -\frac{1}{2}(f' + \bar{f}') = -2\delta\sigma \cdot \frac{1}{2}$

  $z, \bar{z}$: complex coord on $\mathbb{R}^2$ / lightlike coord. in $\mathbb{R}^{1,1}$

  note, in part, that $\partial\bar{\partial}\delta\sigma = 0$, so no trace is induced for $T^\mu{}_\mu$.

- from the above $\delta_f T_{zz} = \underbrace{\frac{c}{24\pi}}_{\text{anomalous transf. law.}} f''' - 2f' T_{zz} - \underbrace{f \partial_z T_{zz}}_{\text{effect of translation}}$ normalization! (-2π)

- in fact, in 2d it is useful to distinguish general conformal transf, param. by arbitrary $f(z)$, $\bar{f}(\bar{z})$ (infinitesimal) from global conformal transf, which exponentiate to well-defined generators on the entire S^2 (Möbius transf)

$$f(z) = \frac{az+b}{cz+d} \quad w/\ ad-bc=1 \qquad SL(2,\mathbb{C}) \simeq SO(3,1)$$

finite! euclidean conf. gp. in $d=2$

(Lorentzian: $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \simeq SO(2,2)$)

(infinitesimally $f_{global} = \{1, z, z^2\}$) note Schwarzian is zero for these

- one may use a basis of f: z^{n+1}, w/ gen. $l_n = -z^{n+1}\partial_z$ (acting on functions). global: l_{-1}, l_0, l_1 (no central ext).

- composing a series of these infinitesimal transf, under a finite conformal transf. transf. as

$$\tilde{T}(\tilde{z}) = \left(\frac{\partial z}{\partial \tilde{z}}\right)^2 \Big[T(z) + \frac{c}{24\pi} \underbrace{\{\tilde{z}, z\}}_{\frac{\tilde{z}'''(z)}{\tilde{z}'} - \frac{3}{2}\left(\frac{\tilde{z}''}{\tilde{z}'}\right)^2} \Big]$$

Schwarzian deriv.

note invar global part of conf. gp

Physical applications / significance of c

- an immediate application of the Schw is deriving the Casimir eng of the CFT on the cylinder.

  vacuum eng. in presence of non-triv. bnd. cond.

- we have $z_{pe} = e^{\frac{w_{cye}}{R}}$ $\quad w \to \tau + i\sigma$

$$T_{cye}(w) = \left(z^2 T_{plane} + \frac{c}{2\cdot 24\pi} \right)\frac{1}{R^2} \qquad \langle T_{pe} \rangle = 0$$

$$H_{cye} = \underset{\text{Wick rot}}{-} \int d\sigma\, T_{\tau\tau}^{cye} = -\int d\sigma\, (T_{ww} + \bar{T}_{\bar{w}\bar{w}}) = -\frac{2\pi R}{R^2}\frac{c}{48\pi}\cdot 2 = -\frac{c}{12R}$$

- the central charge also enters the algebra of the conserved charges.

  - simplest to work on the cylinder, w/ basis of functions $e^{in(\sigma \pm t)}$ (Lorentzian)

  - conserved charges are $L_n = \int_0^{2\pi R} d\sigma\, \underbrace{T_{t+}}_{T_{++} + \underbrace{T_{-+}}_{0}}\, e^{in\overbrace{(t+\sigma)}^{x^+}/R}$

$$\delta_m L_n = \int d\sigma\, \delta_m T_{++}\, e^{inx^+/R} \overset{R}{=} \int d\sigma\, e^{inx^+/R}\Big[-e^{\frac{imx^+}{R}}\partial_+ T_{++} - \frac{2im}{R}e^{\frac{imx^+}{R}} T_{++}$$

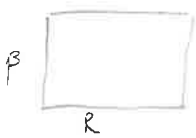
$$+ \frac{c}{24\pi}\left(\frac{im}{R}\right)^3 e^{\frac{imx^+}{R}} \Big] = \int d\sigma\, e^{\frac{i(m+n)x^+}{R}}\left[T_{++}\left(\frac{i}{R}(m+n) - \frac{2im}{R}\right) - \frac{im^3 c}{24\pi R^3} \right]$$

$$= -i(m-n)L_{m+n} - \frac{im^3 c}{12R}\delta_{m+n} = -i[L_m, L_n]$$

$$\Rightarrow \boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n}} \qquad \text{Virasoro algebra}$$

shift: $L_0^{cye} = L_0^{pe} - \frac{c}{24}$

- finally the central charge controls the asymptotic density of states



$$Z_{T^2} = \mathrm{Tr}_{\mathcal{H}_R}\, e^{-\beta H} = Z_\beta(R)$$

Physical applications / significance of c

- an immediate application of the Schw is deriving the Casimir eng of the CFT on the cylinder.

  vacuum eng.

  in presence of non-triv. bnd. cond.

- we have $\quad Z_{pe} = e^{\frac{W_{cyl}}{R}} \quad \xrightarrow{} \tau + i\sigma \quad R$

$$T_{cyl}(w) = \left(z^2 T_{plane} + \frac{c}{2 \cdot 24\pi} \right) \frac{1}{R^2} \qquad \langle T_{pe} \rangle = 0$$

$$H_{cyl} = \underset{\substack{\uparrow \\ \text{Wick rot}}}{-} \int d\sigma \, T_{\tau\tau}^{cyl} = - \int d\sigma \left(T_{ww} + \bar{T}_{\bar{w}\bar{w}} \right) = - \frac{2\pi R}{R^2} \frac{c}{48\pi} \cdot 2 = - \frac{c}{12 R}$$

- the central charge also enters the algebra of the conserved charges.

  - simplest to work on the cylinder, w/ basis of functions $e^{in(\sigma \pm t)}$ (Lorentzian)

  - conserved charges are $\quad L_n = \int_0^{2\pi R} d\sigma \, \underbrace{T_{t+}}_{T_{++} + \underbrace{T_{-+}}_{0}} e^{in \overbrace{(t+\sigma)}^{x^+}/R}$

$$\delta_m L_n = \int d\sigma \, \delta_m T_{++} \, e^{inx^+/R} = \int d\sigma \, e^{inx^+/R} \Big[- e^{\frac{imx^+}{R}} \partial_+ T_{++} - \frac{2im}{R} e^{\frac{imx^+}{R}} T_{++}$$

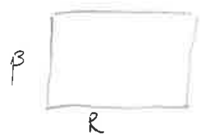
$$+ \frac{c}{24\pi} \left(\frac{im}{R} \right)^3 e^{\frac{imx^+}{R}} \Big] = \int d\sigma \, e^{\frac{i(m+n)x^+}{R}} \Big[T_{++} \left(\frac{i}{R}(m+n) - \frac{2im}{R} \right) - \frac{im^3 c}{24\pi R^3}$$

$$= - i (m-n) L_{m+n} - \frac{i m^3 c}{12 R} \delta_{m+n} = -i [L_m, L_n]$$

$$\Rightarrow \boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m) \delta_{m+n}} \qquad \text{Virasoro algebra}$$

shift $\quad L_0^{cyl} = L_0^{pl} - \frac{c}{24}$

- finally the central charge controls the asymptotic density of states



$$Z_{T^2} = \mathrm{Tr}_{\mathcal{H}_R} e^{-\beta H} = Z_\beta(R)$$

- switching the interpretation of time & space

$$Z_\beta(\cancel{2\pi}R) = Z_{2\pi R}(\beta) \underset{\text{scale invar}}{=} Z_{\cancel{4\pi^2}R^2/\beta}(\cancel{2\pi}R)$$

scale invar

$Z_\beta(R) = Z(\beta/R)$

relates low & high-temperature partition f.

- taking $\beta \to 0$

$$Z_\beta(R) = \underbrace{Z_{\cancel{4\pi^2}R^2/\beta}}_{\infty}(R) \approx e^{-\underbrace{E_0(R)}_{-\frac{c}{12R}}\frac{(\cancel{2\pi}R)^2}{\beta}} = e^{\frac{4\pi^2 R c}{12\beta}} = e^{\frac{\pi^2 c}{3}\frac{R}{\beta}}$$

$$= e^{-\beta F}$$

- thus the free energy is $F(\beta) = -\frac{\pi^2 c}{3}\frac{R}{\beta^2}$ extensive

$$\langle E \rangle = -\partial_\beta \ln Z = \frac{\pi^2 c}{3}\frac{R}{\beta^2} \qquad S = \beta(E - F) = \frac{2\pi^2 c}{3}\frac{R}{\beta}$$

$$\Rightarrow \quad \beta = \sqrt{\frac{\pi^2 c}{3}\frac{R}{E}} \qquad \boxed{S = 2\pi\sqrt{\frac{c}{3} R E}}$$

Cardy's formula

We are now ready to perform our 1^{st} few checks of the AdS/CFT corresp, following our derivation of an AdS_5/CFT_4 corresp. a few lectures back & the general intro to properties of CFTs.

The most basic check is that the symmetries on the two sides of the AdS_{d+1}/CFT_d correspondence are the same.

There are 2 levels @ which one may discuss the symm.

- symmetries of the vacuum state $\leftrightarrow$ isometries of empty AdS
- symmetries of the theory $\leftrightarrow$ asymptotic symm. of a AdS spt. (asympt.)

Anti-de Sitter spacetimes

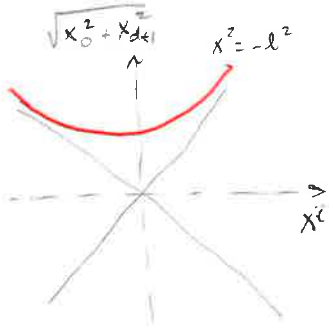
AdS_{d+1} : $\underbrace{\text{maximally symm.}}_{\text{max. \# Killing vect. } \frac{(d+1)(d+2)}{2}}$ spt. of constant negative curvature

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

- isometry gp $\underbrace{SO(d,2)}_{\text{same as "Lorentz" gp in } \mathbb{R}^{d,2}}$ (for Lorentzian AdS_{d+1}), as computed from K.V. alg. (& d-dim'l conformal gp.)

- simplest construction: embedding into $\mathbb{R}^{d,2}$, w/ metric

$$ds^2 = \tilde{\eta}_{MN}\, dx^M dx^N \qquad \tilde{\eta}_{MN} = \begin{pmatrix} - & & & \\ & + & & \\ & & \ddots & \\ & & & + \\ & & & & - \end{pmatrix}$$



* restrict to hyperboloid $\tilde{\eta}_{MN} X^M X^N = -\ell^2$

 - manifestly $SO(d,2)$ preserving constr.

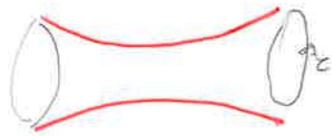
• a metric on AdS_{d+1} can be obtained by writing down an explicit coord. system that solves this constraint, e.g.

global coord : $\begin{cases} X^0 = \ell \cosh\rho \cos\tau & X^i = \ell \sinh\rho\, \Omega_i \\ X^{d+1} = \ell \cosh\rho \sin\tau & \text{w/ } \sum_{i=1}^{d} \Omega_i^2 = 1 \end{cases}$

$\rho \in (0,\infty) \quad \tau \in (0, 2\pi)$

$\Rightarrow ds^2 = \ell^2 \Big(-\cosh^2\rho\, \underset{SO(2)\to\mathbb{R}}{d\tau^2} + d\rho^2 + \sinh^2\rho\, \underbrace{d\Omega^2_{d-1}}_{\text{unit } S^{d-1}} \Big)$

manifest $SO(d)$



after decompactifying the τ coord.

• the isometry gen. are $J_{MN} = -i\left(X_M \frac{\partial}{\partial X^N} - X_N \frac{\partial}{\partial X^M} \right)$ & are identif. w/ the conformal gen. as $J_{0,d+1} = -D \quad J_{\mu\nu} = M_{\mu\nu} \quad J_{\mu 0} = \frac{1}{2}(P_\mu + K_\mu) \quad J_{\mu,d+1} = \frac{i}{2}(P_\mu - K_\mu)$

• the quadratic Casimir $\frac{1}{2} J_{MN} J^{MN}$ $\leftrightarrow$ identified w/ the AdS Laplacian.
(only isometry-invar 2nd order diff. op.)

later

$$\frac{1}{2} J_{AB} J^{AB} \underset{\text{scalar}}{\phi} = -\Big(X^2 \underset{\mathbb{R}^{d,2}\text{ Lapl.}}{\partial_X^2} + X\cdot\partial_X\, (d + X\cdot\partial_X)\Big]\phi$$

• $\mathbb{R}^{d,2}$ ($\text{Mink}^{d+1,1}$) can be foliated by (euclidean) AdS_{d+1} slices as

$(X\partial_X = \ell\partial_\ell)$

$$d\tilde{s}^2 = -d\ell^2 + \ell^2 \overset{\text{rescaled}}{ds^2_{AdS_{d+1}}} \quad , \text{ w/ Laplacian } \partial_X^2 = -\partial_\ell^2 - \frac{d+1}{\ell}\partial_\ell + \underset{\text{not rescaled}}{\Box_{AdS}}$$

thus $\frac{1}{2} J_{AB} J^{AB} \Phi = \Big[-\cancel{\ell^2 \partial_\ell^2} - \cancel{(d+1)\ell\partial_\ell} + \ell^2 \Box_{AdS} + \cancel{(d+1)\ell\partial_\ell} + \cancel{\ell^2\partial_\ell^2} \Big)\phi = \ell^2 \Box_{AdS}\, \phi$

later

· for a **free scalar** $(\Box_{AdS} - m^2)\phi = 0 \Rightarrow \frac{1}{2} J_{AB} J^{AB}\phi = m^2 \ell^2 \phi$

Casimir : $\Delta(\Delta - d)$

$\Rightarrow$ free scalar field in AdS $\subset$ conf. family of a primary operator of dim $\Delta \ni \Delta(\Delta-d) = m^2\ell^2$

· AdS spt. is **non-compact** $\to$ how can we represent it?

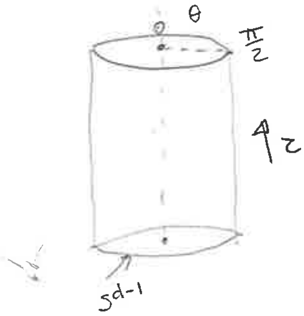
tool : conformal compactif : given (M, g) w/ M non compact, a conformal compactif is a choice of metric, $\tilde{g}$ on M w/ $\tilde{g} = \Omega^2 g \ni (M, \tilde{g})$ can be embedded into a compact domain $\tilde{M}$ of some (other) manifold.

$\partial \tilde{M}$ = **conformal ∞. / bnd.** (want to bring ∞ to finite dist.)

· in practice, we choose coord on M w/ finite ranges $\cosh\rho = \frac{1}{\cos\theta} \in (1,\infty) \Rightarrow \theta \in (0, \frac{\pi}{2})$

$$ds^2 = \frac{l^2}{\cos^2\theta}\left(-d\tau^2 + \underbrace{d\theta^2 + \sin^2\theta \, d\Omega^2_{d-1}}_{\text{metric on } 1/2 \; S^d}\right)$$

(disk)



$$d\tilde{s}^2 = \cos^2\theta \, ds^2 = \mathbb{R}_t \times \text{disk}.$$

· $\Rightarrow$ **Penrose diagram** of AdS_{d+1} = ∞ **solid cylinder** w/ boundary $S^{d-1} \times \mathbb{R}$

· since $\tilde{g} = \Omega^2 g$, the **causal str.** of $\tilde{M}$ is the same as that of M.

Exercise i): Show that lightrays shot radially outwards from the center of AdS take a **finite coord. time** to reach the bnd. Compute it.

- thus, if one puts **reflecting bnd. cond.** for lightrays @ the bnd, they come back in finite coord time $(\pi \cdot \ell)$

· note, however, that the proper dist. to the bnd along a spacelike geodesic is infinite $\int_0^{\pi/2} \frac{d\theta}{\cos\theta}$

Exercise ii) Show that massive particle's geodesics never reach the bnd
(assume for simplicity that the motion is just in the radial plane $d\Omega = 0$)

- ∂_t = Killing vect $\Rightarrow$ cons. quantity $E = -\xi^\mu g_{\mu\nu} \frac{dx^\nu}{d\tau} = \frac{\ell^2}{\cos^2\theta}\frac{dt}{d\tau}$ ($\xi = \partial_t$; τ: proper time; E: eng. part. measured by obs @ $\theta = 0$; ℓ^2: drop for dim'l reasons)

$$ds^2 = -d\tau^2 = \frac{\ell^2}{\cos^2\theta}\left(d\theta^2 - dt^2\right) \Rightarrow \frac{\ell^2}{\cos^2\theta}\left(\frac{d\theta}{d\tau}\right)^2 = \frac{\ell^2}{\cos^2\theta}\left(\frac{E^2\cos^2\theta}{\;}\right)^2 - 1$$
$$= \ell^2 E^2 \cos^2\theta - 1 > 0$$

$\Rightarrow$ $\cos\theta$ cannot become too small $(\cos\theta)_{min} = \frac{1}{\ell^2 E^2} \Rightarrow \theta_{max}$
$= \frac{1}{ch\,\rho_{max}}$

$\Rightarrow$ AdS behaves as a confining box for massive part., indep. of initial energy.
(w/o breaking $\forall$ symm!)

- conseq, Dirichlet bnd. cond. @ the AdS bnd are natural for massive fields, whereas they can be imposed for massless ones.

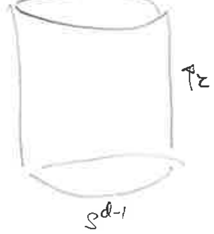
(in AdS, $\exists$ causal fields w/ slightly negative m^2 — more precisely, $\Delta \in \left(\underbrace{\tfrac{d}{2} - 1}_{\text{unit. bnd}}, \tfrac{d}{2}\right)$ for which either Dirichlet or Neumann bnd. cond are possible.

BF bound $\Rightarrow$ map to. $-\psi'' + \left(\vec{k}^2 + \frac{1}{z^2}\left(m^2 - \frac{1-d^2}{4}\right)\right)\psi = \omega^2\psi$ (α; + normalizable, $\alpha < -\frac{1}{4}$ norm. neg-eng. states (i.e. $\omega^2 - k^2 < 0$ bad) $\Rightarrow m^2 > -\frac{d^2}{4}$)

Let us now go back to discussing the (action of) AdS isometries ($\rightarrow$ pg 2).

- as noted the isometry gp of $AdS_{d+1} \cong$ conformal gp. d-dim.

- note in global coord, $D \leftrightarrow \frac{\partial}{\partial\tau}$ (extension into the bulk of time transl. on the cylinder).



$\Rightarrow$ energy $\leftrightarrow$ conformal dim.
spectrum (in global AdS!)

integer-spaced spectrum!
(in the free approx)

- $\exists$ other useful coord systems on AdS, e.g. Poincaré

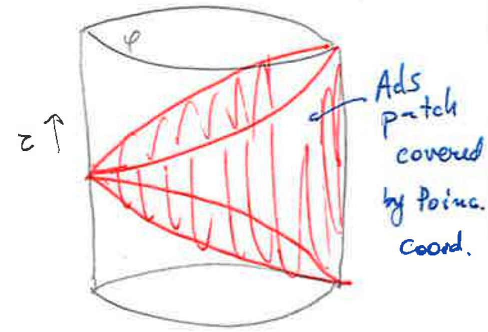
embedding: $X^0 = \dfrac{\ell^2 + x_\mu x^\mu + z^2}{2z}$, $X^i = \ell \dfrac{x^i}{z}$ $i = 1, \dots, d-1$, $X^d = \dfrac{\ell^2 - x_\mu x^\mu - z^2}{2z}$, $X^{d+1} = \ell \dfrac{t}{z}$

$\Rightarrow ds^2 = \dfrac{\ell^2}{z^2}\left(\eta_{\mu\nu}\, dx^\mu dx^\nu + dz^2\right)$ makes Poincaré $\mathbb{R}^{d-1,1}$ & dilatations manifest.

- Poincaré coord only cover a patch of the AdS spt.

  e.g. for $d=2$

$$\begin{cases} X^0 = \ell \cos\tau \cosh\rho = \dfrac{\ell^2 + x^2 - t^2 + z^2}{2z} \\ X^1 = \ell \sinh\rho \sin\varphi = \dfrac{x}{z}\ell \\ X^2 = \ell \sinh\rho \cos\varphi = \dfrac{\ell^2 - x^2 + t^2 - z^2}{2z} \\ X^3 = \ell \sin\tau \cosh\rho = \dfrac{t}{z}\ell \end{cases}$$



$$\tan\tau = \frac{2t\ell}{\ell^2 + x^2 - t^2 + z^2} \qquad \tan\varphi = \frac{2x\ell}{\ell^2 - x^2 + t^2 - z^2} \qquad \text{as } z \to 0 \quad \tan\frac{\varphi \pm \tau}{2} = \frac{x \pm t}{\ell}$$

Minkowski diamond on bnd. cylinder $\Rightarrow$ range of null $\varphi \pm \tau$ is compact. $(-\pi, \pi)$

$-\pi$ π

- can also obtain euclidean global / Poincaré AdS via $\tau \to \tau_E = i\tau$, or the appropriate embedding

- note for Poincaré AdS (eucl), the map is exponential

$$X^0 = \ell \cosh\tau_E \cosh\rho = \frac{\ell^2 + x^2 + t_E^2 + z^2}{2z} \qquad X^1 = \ell \sinh\rho \sin\varphi = \frac{t_E}{z}\ell$$

$$X^3 = \ell \sinh\tau_E \cosh\rho = \frac{\ell^2 - x^2 - t_E^2 + z^2}{2z} \qquad X^2 = \ell \sinh\rho \cos\varphi = \frac{x}{z}\ell$$

(more symm. identif. b/c both X^3 & X^2 are spacelike now)

$$\ell e^{\tau_E} \cosh\rho = \frac{\ell^2}{z} \quad \Rightarrow \quad t_E = \ell e^{-\tau_E} \tanh\rho \sin\varphi$$

$$x = \ell e^{-\tau_E} \tanh\rho \cos\varphi$$

cover all.

• finally, we have the following coord system on eucl. AdS only

$$X^0 = \ell \cosh \rho$$

$$X^I = \ell \sinh \rho \, \Omega_I \quad (I = 1, \dots, d+1)$$

$$\Rightarrow \quad ds^2 = \ell^2 \left(d\rho^2 + \sinh^2\rho \, \underbrace{d\Omega_d^2}_{\text{unit } S^d} \right)$$

spt. looks like a hyperbolic ball ($SO(d+1)$ manifest)

• conformal compactif: let $r = \tanh \frac{\rho}{2} \in (0,1)$, then $ds^2 = \frac{4}{(1-r^2)^2} \big(\underbrace{dr^2 + r^2 d\Omega_d^2}_{\text{ball of radius 1 in flat sp.}} \big)$