Introduction to holography - Lecture VII

Last time : derivation of a particular instance of the AdS/CFT corresp. from string theory, (namely b/w $\mathcal{N}=4$ $SU(N)$ SYM & type IIB string th. on $AdS_5 \times S^5$) , strong-weak , hard to check.

Next few times : general introduction to CFT_d's & some of their (generic) properties (symmetries, operators, str. of corr. f, etc.) in view of a match w/ prop. of gravitational th. in AdS_{d+1} space-times structural

CFTs : relevant in many areas of physics where the behaviour of the system becomes scale-invar:

- continuous (2nd order) phase transitions & quantum critical points
- IR behaviour in QFT (appearance in AdS/CFT)
- UV -11-
- worldsheet string theory (CFT_2)

· oftentimes, theories invariant under scaling are also invar. under the larger gp of conformal transformations : coordinate transf that leave the metric invar. up to a (possibly space-time dependent) overall rescaling

· concretely, under $x^\mu \to x^\mu + \xi^\mu$, $\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \propto g_{\mu\nu}$ =>

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \frac{2}{d} g_{\mu\nu} \nabla_\lambda \xi^\lambda \qquad \text{conformal Killing vect.}$$

· let us fix the metric to be Minkowski : $\eta_{\mu\nu}$ (Lorentzian signature) or $\delta_{\mu\nu}$ (Euclidean)

· denoting the RHS as $f(x) = \frac{2}{d} \nabla_\lambda \xi^\lambda$, we have $2 \nabla_\mu \nabla_\nu \xi_\rho = g_{\mu\rho} \nabla_\nu f + g_{\nu\rho} \nabla_\mu f - g_{\mu\nu} \nabla_\rho f$

$$\Rightarrow \quad d \nabla_\mu \nabla_\nu f = 2 \nabla_\mu \nabla_\nu f - g_{\mu\nu} \Box f$$

• 2 cases: $d>2$, then $(d-2)\nabla_\mu\nabla_\nu f = -g_{\mu\nu}\Box f$, w/ sol'n $f(x) = 2\lambda + 4 b_\mu x^\mu = \frac{2}{d}\partial_\lambda \xi^\lambda$

antisym

$$\Rightarrow \xi^\mu = \underbrace{a^\mu}_{\substack{\text{translations}\\ d}} + \underbrace{\omega^{\mu\nu} x_\nu}_{\substack{\text{Lorentz (L)}\\ \frac{d(d-1)}{2}\\ \text{rotation (E)}}} + \underbrace{\lambda x^\mu}_{\substack{\text{scale}\\ 1}} - \underbrace{(b^\mu x^2 - 2x^\mu b_\nu x^\nu)}_{\substack{\text{special conformal}\\ d}}$$

– total of $\frac{1}{2}(d+1)(d+2)$ transf.

• if $d=2$, then we simply have $\Box f = 0$ — $\partial_z\partial_{\bar z} f = 0$ (Eucl. complex coord) / $\partial_+\partial_- f = 0$ (Lorentzian null).

the general sol'n is $f = \underset{x^+}{f(z)} + \underset{x^-}{\bar f(\bar z)}$ — ∞ dim'l.

• finite conformal transf: $\omega_{\mu\nu} \to$ Lorentz $\Lambda^T \eta \Lambda = \eta$ or rotations $M^T M = \mathbb{1}$

b_μ: special conformal transf.

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$$

• this can be understood as the composition of an inversion $\left(x^\mu \to x'^\mu = \frac{x^\mu}{x^2}\right)$ + translation + another inversion

$$x^\mu \to \frac{\frac{x^\mu}{x^2} - b^\mu}{\left(\frac{x^\mu}{x^2} - b^\mu\right)^2} = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$$

• note. the SC transf. is continuously conn. to the identity, whereas the inversion is not. (the CFT need not have it as a symm.)

• can think of SC as a transf. that fixes the origin ($x^\mu = 0$) & moves ∞, in contradistinction w/ transl, which fix ∞ & move the origin

• all in all, a finite conf. transf. takes the form $\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x)\, \Lambda^\mu{}_\nu(x)$ — position-dep. rescaling & rotations.

$$(x'-y')^2 = \Omega(x)\,\Omega(y)\,(x-y)^2$$

Ω appears in metric transf. $\eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} = \Omega^2(x)\, \eta_{\alpha\beta}$ (angle-preserving transf.).

• the conformal algebra is $SO(d,2)$ (Lorentzian), or $SO(d+1,1)$ in eucl.

(Shiraz 9712074)

Poincaré $\left\{ [M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \right.$ $\qquad [M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho}$

Lie bracket vector fields

$-x^\lambda \partial_\lambda$

K_μ vector ? SC $\left\{ [M_{\mu\nu}, K_\rho] = \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu \right.$ $\qquad [D, P_\mu] = P_\mu \qquad [D, K_\mu] = -K_\mu$

$(K_\mu)^\delta = -2x^\delta x^\mu + x^2 \eta^{\mu\delta}$

raising / lowering for D eigenval

$$[K_\mu, P_\nu] = -2\eta_{\mu\nu} D + 2M_{\mu\nu}$$

standard $[\ ,\ ] = -i[\ ,\ ]_{LB}$. (multiply all gen by $-i$) hermitean $\qquad \mu \in \{0, \dots, d-1\}$

• defining $\quad J_{\mu\nu} = M_{\mu\nu} \quad J_{-1,d} = D \quad J_{\mu,-1} = \frac{1}{2}(P_\mu + K_\mu) \quad J_{\mu d} = \frac{1}{2}(P_\mu - K_\mu)$

they satisfy $SO(d,2)$, w/ the $(-1$ directions $(-1,0)$. (also works in eucl)

• quad. Casimir $\quad C_2 = \frac{1}{2} J^{AB} J_{AB}$. $\quad$ (also quartic) $= \hat{D}^2 - \frac{1}{2}(\hat{K}_\mu \hat{P}^\mu + \hat{P}_\mu \hat{K}^\mu) + \frac{1}{2}\hat{L}_{\mu\nu}\hat{L}^{\mu\nu}$ w/ slightly ≠ signs)

• interesting to define : $\quad M'_{pq} = J_{pq} \qquad p, q \in \{1, \dots, d\}$

$D' = i J_{-1\,0} \qquad P'_p = J_{p,-1} + i J_{p,0} \qquad K'_p = J_{p,-1} - i J_{p,0}$

w/o this for standard algebra.

these new generators satisfy the $SO(d+1,1)$ alg of the euclidean conformal gp.
however, their hermiticity properties are $\quad M'^\dagger = M'$, $D'^\dagger = -D'$ w/ the i; w/o the i it's Hermitean
$P'^\dagger = K'$, $K'^\dagger = P'$

• unitary reps are ∞-dim'l, char. by reps of its maximal compact subgp
$SO(2) \times SO(d)$ $\quad$ (gen by $J_{0,-1} \sim P_0 + K_0$ & J_{pq})

• would like to understand how the conformal gen. act on local operators O_α

$$O_\alpha(x) = e^{i x^\mu P_\mu} O_\alpha(0)\, e^{-i x^\mu P_\mu}$$

true in any quantum QFT.

• the action of the conf. gen on $O_\alpha(x)$ is entirely det. by their action on $O_\alpha(0)$

$$[G, O_\alpha(x)] = e^{i x^\mu P_\mu} [\overbrace{e^{-i x^\mu P_\mu} G\, e^{i x^\mu P_\mu}}^{\tilde{G}}, O_\alpha(0)]\, e^{-i x^\mu P_\mu}$$

generators

$$G - i x^\mu [P_\mu, G] + \frac{i^2}{2} x^\mu x^\nu [P_\mu [P_\nu, G]] + \dots$$

Conformal algebra

- from the CKV $\Rightarrow$ $SO(d,2)$ conformal algebra $[\,,\,] = -i[\,,\,]_{i.B.}$

$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) \quad ..etc. \qquad [K_\mu, P_\nu] = +2i\eta_{\mu\nu}D - 2iM_{\mu\nu}$$

$$[D, P_\mu] = -iP_\mu \qquad [D, K_\mu] = iK_\mu. \qquad \mu = \{0, \ldots, d-1\}$$

$-- = \{0,-1\}$

- can relate these to $SO(d,2)$ generators as $J_{\mu\nu} = M_{\mu\nu}$, $J_{-1,d} = D$, $J_{\mu,-1} = \frac{1}{2}(P_\mu + K_\mu)$.

$J_{\mu d} = \frac{1}{2}(P_\mu - K_\mu)$.

they are all hermitean.

- can now define generators of the euclidean conformal gp. $SO(d+1,1)$ which do not have standard hermiticity relations. As a first step, we introduce

$$M'_{pq} = J_{pq} \qquad P'_p = J_{p,-1} + iJ_{p,0} \qquad K'_p = J_{p,-1} - iJ_{p,0} \qquad D' = J_{0,-1}.$$

(J Lorentzian & hermitean)

which satisfy $[D', P'_p] = P'_p$ $\quad [D', K'_p] = -K'_p$ ← raise/lower for D' eigenval.

$$[K'_p, P'_q] = -2iJ_{pq} + 2\underbrace{J_{0,-1}}_{D'}\delta_{pq}$$

in agreement w/ (2.28) in João's notes.

(I assume the rest work out)

- note $(K')^+ = P'$ & $(D')^+ = D'$, which is consistent w/ the above comm. rels.

- by further letting $D' = -i\underbrace{S_{0,d+1}}_{-1 \text{ for us}}$ $\quad M'_{pq} = S_{pq}$ $\quad P'_p = S_{p0} - S_{p,d+1}$

$K'_p = S_{p0} + S_{p,d+1}$

then the S_{MN} satisfy the euclidean conf. alg (w/ - dir $d+1$, presumably)

•

• luckily, these nested commutators terminate $(\tilde{D}' = D' + i x^\mu P_\mu)$

$$\tilde{D} = D + x^\mu P_\mu \qquad \tilde{M}_{\mu\nu} = M_{\mu\nu} + x_\nu P_\mu - x_\mu P_\nu$$

$$\tilde{K}_\mu = K_\mu - 2x_\mu D + 2M_{\mu\rho}x^\rho - 2x_\mu x^\lambda P_\lambda + x^2 P_\mu$$

⇒ the action of conf. gen on $O_\alpha(x)$ is determined by the action of $D, M_{\mu\nu}, K_\mu$ (stabilizer $x=0$) on $O_\alpha(0)$.

$$[D, O_\alpha(0)] = \Delta^\alpha{}_\beta O_\beta(0) \qquad [M_{\mu\nu}, O_\alpha(0)] = (S_{\mu\nu})_\alpha{}^\beta O_\beta(0) \qquad [K_\mu, O_\alpha(0)] = (\kappa_\mu)^\alpha{}_\beta O_\beta(0)$$

$-i\Delta$

– natural to **diagonalize dilatations** ⟼ scaling ops $e^{i\alpha D} O(0) e^{-i\alpha D} = e^{+\alpha\Delta} O(0)$

$O'(x') = \lambda^{-\Delta} O(x) \quad x' = \lambda x \quad iD = -x\partial_x \quad (e^{-\alpha})^{(-\Delta)}$

– since K_μ acts as a lowering op. for D, natural to look @ ops annih. by it

⇒ **primary operators** $[K_\mu, O(0)] = 0$

• if we act w/ P_μ ⇒ **descendants** (∞ # of these).

• particularizing to ~~such ops.~~ primaries we have

$$[P_\mu, O(x)] = -i\partial_\mu O(x) \qquad [D, O(x)] = -i\Delta O(x) - i x^\mu \partial_\mu O(x)$$

$$[K_\mu, O(x)] = 2i\Delta O(x) x_\mu + 2x^\rho (S_{\mu\rho} O(x)) - 2x_\mu x^\lambda(-i\partial_\lambda O(x)) + x^2(-i\partial_\mu O(x))$$

$$[M_{\mu\nu}, O(x)] = (S_{\mu\nu} O(x)) - i x_\nu \partial_\mu O(x) + i x_\mu \partial_\nu O(x)$$

• these impose stringent constraints on the corr. f. of primary ops. Using

$$0 = \langle 0|[G, O_1 \dots O_n]|0\rangle = \sum_i \langle 0|O_1 \dots [G, O_i] \dots O_n|0\rangle$$

in some quantity (exact.)

we have, e.g. for a scalar 2p.f. $\langle O(x) O'(y)\rangle = f(x-y)$

$$(\Delta_1 + \Delta_2) f + (x^\mu \partial_{x^\mu} + y^\mu \partial_{y^\mu}) f = 0 \quad \Rightarrow \quad f = \frac{1}{|x-y|^{\Delta_1+\Delta_2}} \quad \overset{\Delta_1 = \Delta_2}{\Leftarrow}$$

$$2\Delta_1 x^\mu + 2\Delta_2 y^\mu + 2x_\mu x^\lambda \partial_{x^\lambda} f - x^2 \partial_{x^\mu} f + 2y_\mu y^\lambda \partial_{y^\lambda} f - y^2 \partial_{y^\mu} f = (\Delta_1 - \Delta_2)\frac{(x^\mu - y^\mu)}{(x-y)^{\Delta_1+\Delta_2}}$$

exercise

- the above infinitesimal transf. of primary ops. under conformal transf. result in the following finite transf $\mathcal{U}\, O^{\alpha}(x)\, \mathcal{U}^{-1} = \Omega(x')^{\Delta}\, \underbrace{D(R(x'))}_{\text{rep. } \mathfrak{r}}{}^{\alpha}{}_{\beta}\, O^{\beta}(x')$

  where the finite conf. transf is $\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Omega(x)\, R^{\mu}{}_{\nu}(x)$ (or $\Omega(x')\, R^{\mu}{}_{\nu}(x')$)

  - good composition prop.

$\Rightarrow$ $\langle O_1^{\alpha_1}(x_1) \ldots O_n^{\alpha_n}(x_n)\rangle = \Omega(x_1')^{\Delta_1} \ldots \Omega(x_n')^{\Delta_n} \langle D(R(x_1'))^{\alpha_1}{}_{\beta_1} O^{\beta_1}(x_1') \ldots D(R(x_n'))^{\alpha_n}{}_{\beta_n} O^{\beta_n}(x_n')\rangle$

- for example, the 2-pct. f. of scalar ops must transform as

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \Omega^{\Delta_1}(x_1')\, \Omega^{\Delta_2}(x_2')\, \langle O_1(x_1')\, O_2(x_2')\rangle \quad *$$

  - trivial to see $\frac{1}{(x_1 - x_2)^{\Delta_1+\Delta_2}}$ transf. correctly under transl. $(\Omega = 1)$ rotations $(\Omega = 1)$, and dilatations. $\Omega = \lambda$ $(x'^{\mu} = \lambda x^{\mu})$

  - to check the special conformal transf, sufficient to check transf. under inversions $x^{\mu} \to \frac{x^{\mu}}{x^2}$ w/ $\Omega = \frac{1}{x^2} = x'^2$. Note that, under an inversion $(x_1 - x_2)^2 = \frac{(x_1' - x_2')^2}{\Omega(x_1)\,\Omega(x_2)}$

- then $\frac{1}{|x_{12}|^{\Delta_1+\Delta_2}} = \Omega(x_1')^{\frac{\Delta_1+\Delta_2}{2}}\, \Omega(x_2')^{\frac{\Delta_1+\Delta_2}{2}}\, \frac{1}{|x_{12}'|^{\Delta_1+\Delta_2}}$ same as $*$ if $\Delta_1 = \Delta_2$

- 3-point functions of scalar ops are similarly entirely fixed by conf. symm., up to an overall 3p.f. coefficient

$$\langle O_1(x_1) O_2(x_2) O_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}\,|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}\,|x_{31}|^{\Delta_1+\Delta_3-\Delta_2}}$$

meaningful after norm. the 2pf. to δ_{ij}

- 4-point functions are not fixed by conformal symm b/c one can construct conformal invar. w/ 4 points (from n pts, will have $\frac{n(n-3)}{2}$ invars).

$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} \qquad v = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2}$$

- use a transl. to place $x_1 = (0, \ldots, 0)$
- use a rotation / boost* to set $x_2 = (0, \ldots, 0, 1)$ — *assume spacelike sep.
- use an orthogonal rot to place $x_3 = (0, \ldots 0, x_3^{d-1}, x_3^d)$
- use a special conf. transf. to send x_3 to the line joining x_1 & x_2

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2} \Rightarrow x_1'^\mu = x_1^\mu = (0, \ldots, 0)$$

$$x_2'^\mu = \frac{x_2^\mu - b^\mu}{(x_2 - b)^2} \qquad x_3'^\mu = \frac{x_3^\mu - b^\mu x_3^2}{1 - 2b\cdot x_3 + b^2 x_3^2}$$

choosing $b^\mu = b\,\delta^{\mu, d-1}$, we have $-b = \frac{x_3^{d-1} - b\, x_3^2}{x_3^d}$, or $b = \frac{x_3^{d-1}}{x_3^2 - x_3^d}$ where $x_3^2 = (x_3^{d-1})^2 + (x_3^d)^2$

so, all 3 points are in a line

+ rot. + dilatation, can set

$$x_1'' = (0, \ldots, 0) \qquad x_2'' = (0, \ldots, 0, 1) \qquad x_3'' = (0, \ldots, x_3''^d)$$

(an additional SC w/ $b' = (0, \ldots, -x_3''^d)$ places x_3'' @ ∞, while leaving x_2'' on the same line.

- a 4^{th} pt would be lying in the same (2d) plane as other 3. label as $(0, \ldots, z, \bar z)$ ← $x^{d-1} + i x^d$

$P_1 = (0, \ldots, 0)$, $P_2 = (0, \ldots, z, \bar z)$, $P_3 = (0, \ldots, 1)$, $P_4 = (0, \ldots, \infty)$ — label this as P_2

then $u = z\bar z \qquad v = (1-z)(1-\bar z)$

crossing: $A(u,v) = A\left(\frac{u}{v}, \frac{1}{v}\right)$ $1 \leftrightarrow 2$, $3 \leftrightarrow 4$

- the general form of the 4-pt-f is $\langle O_1 \ldots O_4\rangle = \frac{A(u,v)}{(x_{12}^2\, x_{34}^2)^\Delta}$ — same Δ. $= A(v, u)\left(\frac{u}{v}\right)^\Delta$ $1 \leftrightarrow 3$ or $2 \leftrightarrow 4$