Introduction to holography – Lecture V

Last time : 2 ways to relate theories w/ gravity to a th. w/o gravity

- holography (since $S_{BH} \propto A_H$ & $S_{BH} = S_{max}$, then $\forall$ can be described by d.o.f. on surface of region).

- 't Hooft large N expansion of SU(N) (gauge) theories = genus expansion ≈ perturbative string theory ⊃ gravity
  ↖ will show this today

- started classical bosonic string (Nambu-Goto, Polyakov action).

This time : quantization , target space spectrum , low energy effective action, D-branes

The bosonic string spectrum

- as stated, convenient to work w/ the Polyakov action ↙ D scalars from w-sheet persp.

$$S_P[X^\mu, \gamma_{ab}] = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

- symmetries: Poincaré invar. $X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu$ global

  - reparametrization invar. (diffeos) $\sigma^a \to \tilde\sigma^a(\sigma^a)$ $X^\mu(\sigma^a) \to \tilde X^\mu(\tilde\sigma^a) = X^\mu(\sigma^a)$

    $\gamma_{ab} \to \tilde\gamma_{ab} = \frac{\partial\sigma^c}{\partial\tilde\sigma^a}\frac{\partial\sigma^d}{\partial\tilde\sigma^b}\gamma_{cd}$

  - Weyl invar $\gamma_{ab}(\sigma^a) \to \Omega^2(\sigma^a)\gamma_{ab}(\sigma^a)$

    $X^\mu(\sigma) \to X^\mu(\sigma)$. gauge

    arbitrary rescalings of the metric keeping angles fixed.

- the Weyl + reparam. invar can be used to gauge-fix $\gamma_{ab} = \eta_{ab}$ locally.
  (2d metric ↔ 3 indep. comp, can use the 2 diffeos to make $\gamma \propto \eta$, & Weyl to $\propto \to =$)

- the e.o.m are $\Box_\gamma X^\mu = 0$ (D free bosons)

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu = 0 \quad \text{constraints}$$

- choosing $\gamma_{ab} = \eta_{ab}$, the constraints are $\dot X^2 + X'^2 = 0$ $\quad \dot X \cdot X' = 0$

- A feel for them: $\dot X \cdot X' = 0$ lines of $\sigma = \text{const}$ (normal n_σ) $\perp$ lines of $\tau = \text{const}$ (normal n_τ) $h^{\sigma\tau} \sim \dot X \cdot X' = 0$

  - can use left-over gauge freedom $(\sigma^\pm \to f(\sigma^\pm))$ to fix static gauge $X^0 = R\tau = t$
    $(\Rightarrow X^{0\prime} = 0,\ \dot X^0 = R)$. Then $X^\mu = (t, \vec X)$, & the constraints are
    $\dot X \cdot X' = \dot{\vec X} \cdot \vec X'$ velocity of the string $\perp$ string — transverse oscill. only
    $\dot X^2 + X'^2 = -R^2 + \dot{\vec X}^2 + \vec X'^2$ e.g. for open! end pcts of open string $(X'^\mu = 0)$ move @ speed of light! $\left|\frac{d\vec X}{dt}\right| = 1$

- useful to intro lightcone coord on the w-sheet $\sigma^\pm_{\text{RM}} = \tau \pm \sigma$ ($\swarrow$ LM) ⇒ constraints

$$T_{++} = (\partial_+ X)^2 = \partial_+ X^\mu \partial_+ X_\mu = 0 \quad ; \quad T_{--} = \partial_- X^\mu \partial_- X_\mu = 0$$

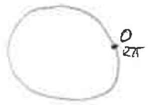
$\approx H(\sigma)$

- the e.o.m is simply $\partial_+ \partial_- X^\mu(\sigma^\pm) = 0$

- for the action to be minimized, we also need the fields to satisfy appropriate bnd. conditions. Several possibilities:

  - closed strings: periodic $X^\mu(\tau, 0) = X^\mu(\tau, 2\pi)$ &

    continuous $\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, 2\pi)$.



  - open strings: Neumann bnd. cond $\partial_\sigma X^\mu(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0$

    endpoint can sit anywhere in target sp.

    - Dirichlet bnd. cond $\delta X^\mu(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0$

      endpoints are @ fixed positions in target sp

- the var. of the action is $\delta S = \frac{1}{2\pi\alpha'}\left[\int d\tau d\sigma \left(\eta^{ab}\partial_a\partial_b X^\mu\right)\delta X_\mu - \int d\tau\, \partial_\sigma X_\mu \delta X^\mu \Big|_{\sigma=0}^{\pi}\right]$ (2π for closed)

- sols to the e.o.m. can be expanded in left/right-moving modes

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \qquad \sigma^\pm = \tau \pm \sigma$$

w/ $X^\mu_{L/R}(\sigma^\pm) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^\pm + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-in\sigma^\pm}$ closed string

(x^μ: C.M. position @ $\tau = 0$; p^μ: C.M. momentum; α^μ_n / $\tilde\alpha^\mu_n$)

(else $x^\mu + \alpha' p^\mu \tau$ ~ rel pt. part.)

reality cond: $(\alpha^\mu_n)^* = \alpha^\mu_{-n}$

- note that $X^\mu_{L,R}$ are not individually periodic in σ, but their sum is ($\sigma^+ + \sigma^- = 2\tau$)

- p^μ: cons. charge assoc. w/ $X^\mu \to X^\mu + \text{const}$ $\quad \int d\sigma \frac{\partial \mathcal{L}}{\partial \dot X_\mu} = \frac{1}{2\pi\alpha'}\int d\sigma\, \dot X^\mu = p^\mu$

- in the open string, the LM & RM osc. get related via the bnd. cond, e.g. for Neumann $\alpha^\mu_n - \tilde\alpha^\mu_n = 0 = e^{-in\pi}\alpha^\mu_n - e^{in\pi}\tilde\alpha^\mu_n \Rightarrow \tilde\alpha^\mu_n = \alpha^\mu_n$

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum \frac{1}{n}\alpha^\mu_n e^{-in\tau}\cos n\sigma$$

- then, we need to take into account the constraints $(\partial_+ X)^2 = (\partial_- X)^2 = 0$

  $\Rightarrow$ non-linear constraints on p^μ, α_m^μ. We compute

$$\partial_+ X^\mu = \partial_+ X_L^\mu = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\sigma^+} \equiv \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in\sigma^+}$$

where we defined $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$

$$\partial_+ X^\mu \cdot \partial_+ X_\mu = \frac{\alpha'}{2} \sum_{m,p} \alpha_m^\mu \cdot (\alpha_p)_\mu \, e^{-i\underbrace{(m+p)}_{n}\sigma^+} = \frac{\alpha'}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m} \, e^{-in\sigma^+}$$

$$\equiv \alpha' \sum_n L_n e^{-in\sigma^+} = 0 \quad \text{w/} \quad \boxed{L_n \equiv \frac{1}{2} \sum_m \alpha_m \cdot \alpha_{n-m}}$$

- similarly for the RM modes, we define $\tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$, & the constr. is

$$0 = \partial_- X^\mu \cdot \partial_- X_\mu = \alpha' \sum_n \tilde{L}_n e^{-in\sigma^-} \quad \text{w/} \quad \tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m}$$

$\Rightarrow$ ∞ # of constraints : $L_n = \tilde{L}_n = 0$, $n \in \mathbb{Z}$

- note that $L_0, \tilde{L}_0$ contain $p^2 \propto \tilde{\alpha}_0^2 = \alpha_0^2$ $\rightarrow$ square of CM mom. $= -M^2$ rest mass of Lor. part.

  which appears in both constraints.

  $\Rightarrow$ constraint on the action of the oscillators due to these 2 ≠.

  expr. for the mass (level matching)

- open strings $\rightarrow$ 1 constraint w/ α_0^μ def. as $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$

$\approx H(r)$

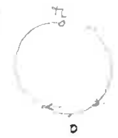
- for a closed string the soln to the e.o.m (periodic $\sigma \in (0, 2\pi)$) can be expanded in left / right - moving modes $X^\mu(\tau,\sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$ $\sigma^\pm = \tau \pm \sigma$

$$X^\mu_{L,R}(\sigma^\pm) = \frac{1}{2} x^\mu + \frac{1}{2}\alpha' p^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^\pm}$$

x^μ: c.M. position @ $\tau=0$; p^μ: c.M. momentum (in target sp.) ; $\tilde{\alpha}_n^\mu$; $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$ reality

- for an open string w/, say, Neumann bnd. cond $\partial_\sigma X^\mu \big|_{\sigma=0,\pi} = 0$ $(\tilde{\alpha}_n^\mu = \alpha_n^\mu)$

(no momentum flowing out endpoints)



$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma$$

p^μ: c.M. momentum ; $\cos n\sigma$: standing waves ; $\sigma \to -\sigma$ refl by Neumann

- factor of 2 due to $\sigma \in (0,\pi)$ for the open string : $p^\mu = -\frac{1}{4\pi\alpha'}\int_0^\pi d\sigma \frac{\partial L}{\partial \dot{X}^\mu} = \frac{1}{2\pi\alpha'}\int_0^\pi \dot{X}^\mu$

- the constraints are, classically

$$\alpha_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu$$

∞ #

$$\begin{cases} T_{++} = \partial_+ X^\mu \partial_+ X_\mu = \alpha' \sum_n L_n e^{-in\sigma^+} = 0 & 0 = \boxed{L_n \equiv \frac{1}{2}\sum_m \alpha_m^\mu (\alpha_{n-m})_\mu} \\ T_{--} = \alpha' \sum_n \tilde{L}_n e^{-in\sigma^-} = 0 & \text{or} \quad \tilde{L}_n = \frac{1}{2}\sum_m \tilde{\alpha}_m^\mu (\tilde{\alpha}_{n-m})_\mu = 0 \qquad \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu \end{cases}$$

- they are conserved, as cls. conformal invar. of $S_P = \int d^2\sigma\, \partial_+ X^\mu \partial_- X_\mu$ $\Rightarrow$ $\int d\sigma f(\sigma^\pm) T_{\pm\pm}$ is conserved.

(residual symm. after fixing diff + Weyl).

various possibilities : old covariant Q (here)
- lightcone (D. Tong. lectures, Polchinski ...)

Quantization

- promote α_n^μ, $\tilde{\alpha}_n^\mu$ to operators w/ canonical comm. rels determined by

$$[X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = i\delta^\mu{}_\nu\, \delta(\sigma-\sigma') \qquad [X^\mu, X^\nu] = [\Pi_\mu, \Pi_\nu] = 0$$

which imply that $[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\,\eta^{\mu\nu}\delta_{m+n,0}$ $\quad [\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu$

2 ∞ towers of oscillators

- α_m^μ, $m>0$ are annihilation operators
- α_{-m}^μ, $m>0$ are creation ops, $\alpha_{-m}^\mu = (\alpha_m^\mu)^\dagger$, in standard norm $\frac{\alpha_{-m}^\mu}{\sqrt{m}}$

- define vacuum state $\quad \alpha_n^\mu |0\rangle = \tilde{\alpha}_n^\mu |0\rangle = 0 \quad , \forall n>0$

$\hat{p}^\mu |0\rangle = p^\mu |0\rangle \quad \Rightarrow$ better notation $|0;p^\mu\rangle$

($\hat{p}^\mu$: operator; p^μ: eigenvalue)

* note $|0;p^\mu\rangle$ is the vacuum state of a single string (not of the spt), which can carry momentum p^μ

- build Fock space of states as in standard QFT
  - excited states $\quad \alpha_{-1}^{\mu_1} \dots \alpha_{-1}^{\mu_n}\, \alpha_{-2}^{\nu_1} \dots |0, p^\mu\rangle$
  - classified into representations of the target sp. Lorentz gp.

* note this Fock sp. contains many negative norm states, due to $[\alpha_m^0, \alpha_n^0] = -m\delta_{m+n}$

- impose the constraints on physical states : $L_m |\phi_{phys}\rangle = \tilde{L}_m |\phi_{phys}\rangle = 0$
  - normal ordering ambiguities (in L_0 only)
  - the L_m do not commute QM $\quad [L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12} m(m^2-1)\delta_{m+n}$

- consequently, we will only require that $L_m|\phi_{phys}\rangle = 0$, $\forall\, m>0$

$(L_0 - a)|\phi_{phys}\rangle = 0$ where
(L_0: N.O.; a: ambiguity)

$$L_0 \equiv \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{n,\mu}$$

+ RM. for the closed string.

- this is sufficient to ensure that $\langle\psi_{phys}|L_{m_1}\dots L_{m_n}|\phi_{phys}\rangle = 0$ $\forall\,\psi_{ph},\phi_{phys}$ & $\forall\, m_i$, not necessarily positive. If all negative ones are to the left of pos. ones, then ok, since $\langle\psi_{phys}|L_{-m} = (L_m|\psi_{phys}\rangle)^\dagger = 0$; note, however there could be c-# commutators that don't vanish*

- note also that $\nexists$ an ordering ambiguity in the target sp angular mom. ops

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu\alpha_n^\nu - \alpha_{-n}^\nu\alpha_n^\mu)$$

& $[J^{\mu\nu}, L_n] = [J^{\mu\nu}, \tilde{L}_n] = 0$ $\Rightarrow$ phys. st. cond. are invariant under Lorentz transf
$\Rightarrow$ phys states will form Lorentz multiplets

(this quantiz. method is known as old covariant quantiz.)

- note that the $(L_0 - a)|\phi_{phys}\rangle = 0$ cond. determines $p^2 = -M^2$ (mass)^2 of the string ≈ looks pointlike from far away ⇔ mass of the associated particle

in terms of the level $N = \sum_n n N_n$ (eigenval. of modified # op $\hat{N} = \sum_{n=1}^{\infty}\alpha_{-n}^\mu\alpha_{n\mu}$ which counts # osc in the state, weighted by n

- for open strings $(L_0 - a)|\phi_{phys}\rangle = (\frac{1}{2}\underbrace{\alpha_0^2}_{2\alpha' p^2} + \hat{N} - a)|\phi_{phys}\rangle = 0 \Rightarrow$ $\boxed{M^2 = \frac{1}{\alpha'}(N-a)}$

- for closed strings, need both $(L_0 - a)|\phi_{phys}\rangle = (\tilde{L}_0 - a)|\phi_{phys}\rangle = 0 \Rightarrow$ $\boxed{M^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-a)}$
$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$

$\Rightarrow N = \tilde{N}$ level matching (cond. that relates otherwise decoupled LM & RM)

determines spacetime spectrum of the string

- in order to det. spt. spectrum, crucial to det N.O. ct. a

- should also ensure constraints remove the negative norm states ($\approx$ sufficient, as these states are generated by α^0_{-m}, & constr $L_m \sim \alpha_0^\mu \cdot \alpha_{m,\mu} + O(\alpha^2)$ can in principle remove precisely α^0_m for timelike p^μ (with $\alpha_0^\mu \sim p^\mu$)

- this (tedious) exercise can be done → need $a = 1$ and $D = 26$
  ↳ can also get this from comm. rels in lightcone gauge (Tong)
  (more precisely, requiring no negative norm st. in OCQ leads to either $a = 1$, $D = 26$ (w/ tons of zero norm states that decouple) or $a \leq 1$, $D \leq 25$. To see inconsistency w/ the latter option, one should go to 1-loop) see Green, Schwarz, Witten

- the necessity for this part. choice is easier to see in the (non-covariant) lightcone gauge, $\tau = X^+$ (the reparam. freedom left over in conformal gauge allows setting τ equal to any function that satisfies the free wave eqn., in part X^μ). The constraints then fully det X^- in terms of X^i, $i = 1, \dots, D-2$. At level 1 (open string) $\exists$ $D-2$ states → not Lorentz multiplet unless $M^2 = 0$.

The string spectrum

- open string $M^2 = \frac{1}{\alpha'}(N - a)$, with $a = 1$

level	state	mass²	
$N=0$	$\|0;p^\mu\rangle$	$M^2 = -\frac{1}{\alpha'}$	tachyon (unstable vac. → ignore (susy))
$N=1$	$\alpha^\mu_{-1}\|0,p^\mu\rangle$	$M^2 = 0$	massless vector
$N>1$	∞ tower of equally spaced massive states		

– Lorentz mult $J_{max} \sim N = \alpha' M^2 + 1$
Regge trajectory

- to see the state w/ $M^2 = 0$ $|\psi\rangle = \zeta_\mu \alpha^\mu_{-1} |0, p^\mu\rangle$ is indeed a gauge field
  note the phys. state cond $L_1 |\psi\rangle = 0$ ⇒ $(\alpha_0^\nu (\alpha_1)_\nu + \sum_{m \neq 0} \alpha_{-m} \cdot \alpha_{m+1}) |\psi\rangle = 0$
  ⇒ $p^\mu \zeta_\mu = 0$

- the norm of these st. is $\langle\psi|\psi\rangle = \zeta^2$ <
  $D-2$ positive norm st. $\perp p^\mu$
  $\zeta^\mu \propto p^\mu$ zero norm state decouples.

- adding to it a spurious st. (⊥ all the phys.) of the form $L_{-1}|0, p^\mu\rangle = p_\mu \alpha^\mu_{-1} |0, p\rangle$, one can make $\zeta^\mu \to \zeta^\mu + \lambda p^\mu$ (gauge inv.)

- closed strings

$$M^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-a)$$

level	state	mass^2	
$N=\tilde{N}=0$	$\|0,p^\mu\rangle$	$M^2 = -\frac{4}{\alpha'}$	tachyon
$N=\tilde{N}=1$	$\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu \|0,p^\mu\rangle$	$M^2=0$	(*)
$N=\tilde{N}>1$	∞ tower of equally spaced massive states → Lorentz mult.		

- (*) decomposing $\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0,p\rangle$ in terms of $SO(D-1,1)$ reps.

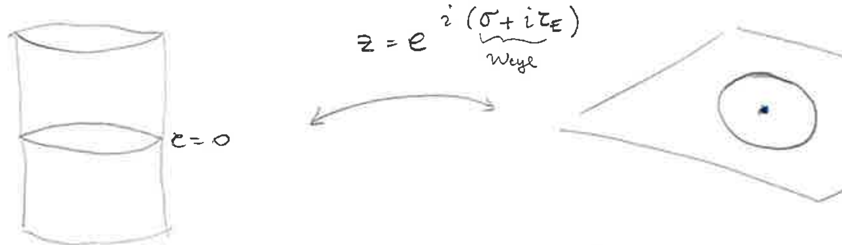
  - $g_{\mu\nu}$ symmetric traceless spin 2 graviton ! string. th. contains gravity
    (gauge symm visible as above)

  - $B_{\mu\nu}$ anti-symm. "B-field" 2-form gauge field coupling to the string $q\int d\tau A_\mu \dot{X}^\mu \overset{\text{pcl part.}}{\longleftrightarrow}_{\text{analogy}} \int d\tau d\sigma\, B_{\mu\nu}\, \varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu$

    gauge invar $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$

  - Φ trace : $\alpha_{-1}^\mu (\tilde{\alpha}_{-1})_\mu |0,p\rangle$ spin 0 dilaton

- note all these are states of the free boson CFT on the cylinder. In such th. $\exists$ a state - operator correspondence whereby ($\sigma^\pm \to f(\sigma^\pm)$) state on cyl $\leftrightarrow$ local op. on plane



$ds^2 = dr^2 + r^2 d\sigma^2 = r^2\left(\frac{dr^2}{r^2} + d\sigma^2\right)$

vertex operator

- in this case, $g_{\mu\nu} \leftrightarrow h_{\mu\nu}\, \partial_a X^\mu \partial^a X^\nu e^{ip\cdot X}$

- exponentiating this ($\approx$ coherent st. of gravitons) $\Rightarrow S_p = \int d^2\sigma\, \eta_{\mu\nu}\partial_a X^\mu \partial^a X^\nu \to$

  $\to S_p + \int d^2\sigma\, h_{\mu\nu}(X)\, \partial_a X^\mu \partial^a X^\nu \Rightarrow$ the target sp. metric has become dynamical ! the wsheet β function (vanishing) for $G_{\mu\nu}$ (gen backgnd. metric) (∞ # of w-sheet couplings) → Einstein eqn. in target sp ! (from consistency of string!)

- the **dilaton** appears in the w-sheet action as $\int d^2\sigma \sqrt{\gamma}\, R[\gamma]\, \Phi$

note that for $\phi = \phi_0 = \text{const}$, this term is $\propto \phi_0 \chi$ ← Euler char. of the w-sheet

⇒ the **string coupling** that counts topologies / string interactions

$$g_s = \langle e^{\phi_0} \rangle$$

↑ ct. mode, usually as $r \to \infty$ ↖ exp. value of a field in the th. (not an arbitrary param.) (needs to be small for string pert. th. to be valid)

Lessons : from a target sp. perspective, the various excitations of the string ↔ ∞ tower of massive (higher spin) part. w/ M^2 as above (spacing $\propto 1/\alpha'$)

- they all fit into massless / massive reps. of the Lorentz gp.

- @ the massless level, closed strings ⊃ graviton

open strings ⊃ gauge boson $\int_{\partial\Sigma} d\tau\, A_\mu \frac{dX^\mu}{d\tau}$ ~ exp. photon vertex op.

- @ low energies ($E \ll \frac{1}{\sqrt{\alpha'}}$), the massive string modes can be integrated out

$R_c \gg \sqrt{\alpha'}$ needed for approx

⇒ effective action for the massless ones $S_{EH} + S_B + S_\phi$ coupled

$$S_{\substack{\text{closed}\\ \text{bosonic}}} = \frac{1}{2\kappa_0^2} \int d^Dx \sqrt{-G}\; e^{-2\phi} \left[R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4 \partial_\mu \phi\, \partial^\mu \phi + \mathcal{O}(\alpha') \right]$$

$2\kappa_0^2 \sim \alpha'^{\frac{D-2}{2}}$; G: string-frame metric; $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ gauge field strength; $\mathcal{O}(\alpha')$: higher order corr. suppr. by α'

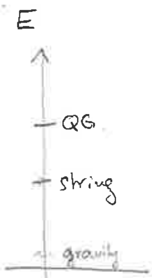
- can redefine the target sp. metric as $\tilde{G}_{\mu\nu} = e^{-\frac{4\phi}{D-2}} G_{\mu\nu}$ Einstein frame metric

∋ the 1st term is standard Einstein - Hilbert.

- effective D-dim'l (D=10) Newton's const $G_D \sim (\alpha')^{\frac{D-2}{2}} g_s^2 \equiv \ell_p^{D-2}$

⇒ two scales : $\ell_s = \sqrt{\alpha'}$ effects of massive string modes

- if $\ell_p \ll \ell_s$ if $g_s \ll 1$ $\ell_p = g_s^{\frac{2}{D-2}} \sqrt{\alpha'}$ quantum gravity effects



pert. string exp $\sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g\left(\frac{\alpha'}{R^2}\right)$

- open strings : massless mode → gauge field A_μ w/ low-eng. eff. action

$$\int d^Dx \left(-\frac{1}{4g_{s,d'}} F_{\mu\nu} F^{\mu\nu} \right)$$

- if we consider instead superstrings (add fermions on w-sheet) ψ^μ
  - ⊕ can project out the tachyons
  - consistent in $D = 10$
  - additional massless fields "RR" (Ramond-Ramond) (p+1)-form gauge fields

    $C_{\mu_1 \dots \mu_p \mu_{p+1}}$ totally antisymm , p even (IIA) / odd IIB

  - + fermions ; has target sp. supersymmetry (low-eng. eff. action highly constrained)

e.g

$$S_{IIB} = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{G} \left\{ e^{-2\Phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H_{\mu\nu\rho}^2 \right) - \frac{1}{2} \left(F_\mu^2 + \frac{1}{3!} \tilde{F}_{\mu\nu\rho}^2 + \frac{\tilde{F}_{\mu\nu\rho\sigma\lambda}^2}{5!} \right) \right\}$$

+ Chern-Simons terms + fermions $\quad F_{\mu_1 \dots \mu_{p+2}} = \partial_{[\mu_1} C_{\mu_2 \dots \mu_{p+2}]}$

  - these RR (p+1)-form gauge fields naturally couple to p-spatial dim'l objects (they do not couple to the string)

- possible to obtain a non-linear generalization of this action valid for constant field strengths ($\partial F \approx 0$) by performing path int on the disk (open string) in presence of a backgnd gauge field

$$Z[F] = \frac{1}{g_s} \int \mathcal{D}X^\mu \, e^{-S_P(X^\mu, A)} = \frac{1}{(4\pi^2\alpha')^5 g_s} \int d^D X_0 \sqrt{\det(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}$$

tree ↗

Born-Infeld action

  - exact in α', but only valid for slowly-varying F
  - non-linear correction to electrodyn. (smoothens out div. near pct-like sources)

    e.g. in $d=4$ $\quad E_r = F_{rt} = \dfrac{Q}{\sqrt{r^4 + (\alpha' Q)^2}}$

- another interesting generalization : add non-dynamical Chan-Paton d.o.f. to the open string endpoints $\lambda^i{}_j$ (i ← fund, j ← antifund) of U(N) → non-abelian Yang-Mills / BI

D-branes

- all discussion so far was perturbative in g_s

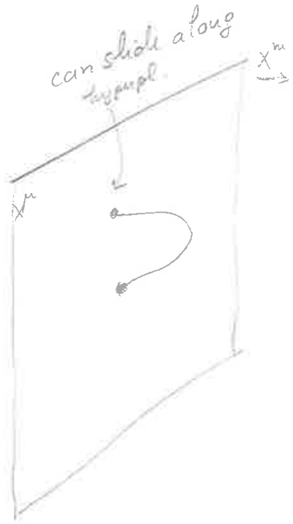
- will now discuss objects that will turn out to be non-perturbative

- they led to huge progress in string th: 1^{st} microscopic explanation of black hole entropy
  $S = \ln \Omega$ (Strominger-Vafa '96)
  - the discovery of AdS/CFT (Maldacena '97)

- importantly, $\exists$ two ways to think about them

(1.) remember open strings can have either Neumann ($\partial_\sigma X^\mu = 0$) or Dirichlet ($\delta X^\mu = 0$) bnd. cond. @ their endpoints

- a Dp-brane "Dirichlet membrane": $p+1$-dim'l hypersurface ($p = \#$ spatial dir) is specified by:



  - Neumann bnd. cond. along the hypersurface

$$\partial_\sigma X^\mu \big|_{\sigma=0,\pi} = 0 \qquad \mu = \{0, \dots, p\}$$

  - Dirichlet bnd. cond in the transverse directions

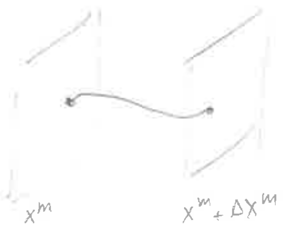
$$\delta X^m \big|_{\sigma=0,\pi} = 0 \qquad m = \{p+1, \dots, D-1\}$$

- mode expansion: $X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma$

$$N: \quad X_L^\mu(\tau,\sigma) = X_R^\mu(\tau,-\sigma)$$

no momentum

$$X^m(\tau,\sigma) = x^m + \frac{\sigma}{\pi} \Delta X^m + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \sin n\sigma$$



string stretched b/w 2 D-branes

$$D: \quad X_L^m(\tau,\sigma) = -X_R^m(\tau,-\sigma).$$

- the mass shell cond. for such a string is $M^2 = -p^\mu p_\mu = \frac{1}{\alpha'}(N-a) + \frac{|\Delta \vec{x}|^2}{(2\pi\alpha')^2}$

  - massless states when endpts on the same D-brane : $\alpha_{-1}^{\mu}|0,p^\mu\rangle$, $\alpha_{-1}^{m}|0,p^\mu\rangle$

    - massless vector A_μ $\mu = \{0,\dots,p\}$ w/ coupling $\int_{\partial\Sigma} d\tau\, A_\mu \frac{dx^\mu}{d\tau}$

    - D-p-1 scalars Φ^m w/ coupling $\int_{\partial\Sigma} \phi^m \partial_\sigma X_m$

  - low eng. eff action $S \sim \int d^{p+1}x \left(-\frac{1}{4}F_{\mu\nu}^2 + (\partial_\mu \phi^m)^2 + \dots\right) \cdot \frac{1}{g_s\, \alpha'^{\frac{p-3}{2}}}$ (along brane)

  - also non-abelian generaliz SU(N) Yang-Mills $S = \frac{1}{g_{YM}^2}\int d^{p+1}x \left[-\frac{1}{4}\mathrm{tr}\, F^2 + \mathrm{Tr}(D_\mu \phi^m)^2 + \mathrm{tr}\,[\phi^m,\phi^n]^2 + \dots\right)$

- a priori, a D-brane is just a hyperplane where open strings can end

- however, it should be considered a fully dynamical object that is necessarily part of open + closed string th. ↑ makes sense in th. of gravity (no rigid objects)

  (this can be seen from T-duality, a symm. of compactified closed (bosonic) string th. under $R \to \frac{\alpha'}{R}$, which is implemented as $X_R \to -X_R$ on the w-sheet : interchanges momentum & winding. For open strings, this interchanges N & D bnd. cond ⇒ D-branes must be part of string th. The gauge field $A_\mu \leftrightarrow \phi^m$ as can be seen from T-dualizing in presence of a non-triv. Wilson line. More general configs → wiggly hyperpl.)

- the scalars ϕ^m should be thought of as the transverse displacement of the Dp-brane ($\phi^m = $ const ⇒ $\delta X^m = \phi_0^m$)

- this is similar to the closed string case, where a massless closed string excitation ($h_{\mu\nu}\, \alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0,p\rangle$) about flat target sp. corresp. to a fluctuation of the target sp. geom.

- here, a massless open string state ↔ fluctuation of the hypersurf ⇒ D-branes are dynamical obj

- the coupling of D-branes to the target sp. fields is

↙ world vol. tens. + conn.

$$S_{DBI} = - \frac{1}{\alpha'^{\frac{p+1}{2}}} \int d^{p+1}x \left\{ e^{-\phi} \sqrt{-\det\left(G_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)} + \dots \right\}$$

↑ induced metric & B-field $\quad G_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N G_{MN}$, etc.

- for slowly-var $F_{\mu\nu}$, T-duality inv.

$T_p = \frac{1}{(2\pi)^p \, \alpha'^{\frac{p+1}{2}}}$ tension very heavy for g_s small

- in superstring theory Dp-branes can be shown to carry RR charge (min quantum).

$\mu_p \int d^{p+1}x \; C^{(p+1)}_{\mu_1 \dots \mu_{p+1}}$ & to preserve ½ supersymmetry (very important for their identification @ strong coupling) – add coupling: $e^{2\pi\alpha' F + B_2} \wedge \sum_k C_k \Big|_{p+1 \text{ form}}$.

- the low-energy effective action = supersymmetric version of SU(N) Yang-Mills

$$S = \frac{1}{g^2_{YM}} \int d^{p+1}x \left[-\frac{1}{4} \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right) + \frac{1}{2} \mathrm{tr}\left(\partial_\mu \phi^m \partial^\mu \phi_m\right) + \mathrm{tr}\left([\phi^m, \phi^n]\right)^2 + \text{fermions} \right]$$

= dimensional reduction of $\mathcal{N}=1$ SYM in 10 d to p+1 dims

$\mathcal{N}=4$ in 4d

- $g^2_{YM} = g_s \alpha'^{\frac{p-3}{2}}$. For $p=3$ the SYM theory is conformally invar (CFT_4) and g_{YM} is an exactly marginal coupling (family of susy CFTs, param by g_{YM} & θ in $\theta \int F_{\mu\nu} \tilde{F}^{\mu\nu}$

↓ $C_0 \, F \wedge F$.

Summary : D-branes are non-perturbative objects in string th., which must be included if open strings are present (T-duality)

- ½ BPS (susic) & charged under the RR p+1-form fields

- described by SYM th. @ low energies $g^2_{YM} = g_s \alpha'^{\frac{p-3}{2}}$ – tractable if the 't Hooft coupling $\lambda = N g^2_{YM}$ is small